The properties of limits of sequences listed in the next theorem parallel those given for limits of functions of a real variable in Section 1.3.

THEOREM 9.2 Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$

2. $\lim_{n\to\infty} ca_n = cL$, c is any real number.

3. $\lim_{n\to\infty} (a_n b_n) = LK$

4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

EXAMPLE 3 Determining Convergence or Divergence

See LarsonCalculus.com for an interactive version of this type of example.

a. Because the sequence $\{a_n\} = \{3 + (-1)^n\}$ has terms

$$2, 4, 2, 4, \ldots$$

See Example 1(a), page 584.

that alternate between 2 and 4, the limit

$$\lim_{n\to\infty} a_n$$

does not exist. So, the sequence diverges.

b. For $\{b_n\} = \left\{\dfrac{n}{1-2n}\right\}$, divide the numerator and denominator by n to obtain

$$\lim_{n\to\infty} \frac{n}{1-2n} = \lim_{n\to\infty} \frac{1}{(1/n) - 2} = -\frac{1}{2}$$

See Example 1(b), page 584.

which implies that the sequence converges to $-\frac{1}{2}$.

EXAMPLE 4 Using L'Hôpital's Rule to Determine Convergence

Show that the sequence whose nth term is $a_n = \dfrac{n^2}{2^n - 1}$ converges.

Solution Consider the function of a real variable

$$f(x) = \frac{x^2}{2^x - 1}.$$

Applying L'Hôpital's Rule twice produces

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{(\ln 2)2^x} = \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0.$$

Because $f(n) = a_n$ for every positive integer, you can apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1} = 0.$$

See Example 1(c), page 584.

So, the sequence converges to 0.

TECHNOLOGY Use a graphing utility to graph the function in Example 4. Notice that as x approaches infinity, the value of the function gets closer and closer to 0. If you have access to a graphing utility that can generate terms of a sequence, try using it to calculate the first 20 terms of the sequence in Example 4. Then view the terms to observe numerically that the sequence converges to 0.

The symbol $n!$ (read "n factorial") is used to simplify some of the formulas developed in this chapter. Let n be a positive integer; then ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n.$$

As a special case, **zero factorial** is defined as $0! = 1$. From this definition, you can see that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, and so on. Factorials follow the same conventions for order of operations as exponents. That is, just as $2x^3$ and $(2x)^3$ imply different orders of operations, $2n!$ and $(2n)!$ imply the orders

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)$$

and

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \cdot (n+1) \cdots 2n$$

respectively.

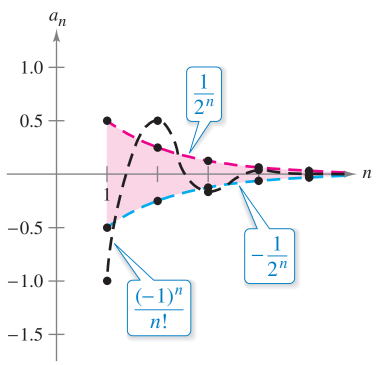
Another useful limit theorem that can be rewritten for sequences is the Squeeze Theorem from Section 1.3.

THEOREM 9.3 Squeeze Theorem for Sequences

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then $\lim_{n\to\infty} c_n = L$.

EXAMPLE 5 Using the Squeeze Theorem

Show that the sequence $\{c_n\} = \left\{(-1)^n \dfrac{1}{n!}\right\}$ converges, and find its limit.



For $n \ge 4$, $(-1)^n/n!$ is squeezed between $-1/2^n$ and $1/2^n$.
Figure 9.2

Solution To apply the Squeeze Theorem, you must find two convergent sequences that can be related to $\{c_n\}$. Two possibilities are $a_n = -1/2^n$ and $b_n = 1/2^n$, both of which converge to 0. By comparing the term $n!$ with 2^n, you can see that

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots n = 24 \cdot \underbrace{5 \cdot 6 \cdots n}_{n-4 \text{ factors}} \qquad (n \ge 4)$$

and

$$2^n = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 16 \cdot \underbrace{2 \cdot 2 \cdots 2}_{n-4 \text{ factors}}. \qquad (n \ge 4)$$

This implies that for $n \ge 4$, $2^n < n!$, and you have

$$\frac{-1}{2^n} \le (-1)^n \frac{1}{n!} \le \frac{1}{2^n}, \quad n \ge 4$$

as shown in Figure 9.2. So, by the Squeeze Theorem, it follows that

$$\lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0.$$

Example 5 suggests something about the rate at which $n!$ increases as $n \to \infty$. As Figure 9.2 suggests, both $1/2^n$ and $1/n!$ approach 0 as $n \to \infty$. Yet $1/n!$ approaches 0 so much faster than $1/2^n$ does that

$$\lim_{n\to\infty} \frac{1/n!}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

In fact, it can be shown that for any fixed number k, $\lim_{n\to\infty} (k^n/n!) = 0$. This means that *the factorial function grows faster than any exponential function*.

In Example 5, the sequence $\{c_n\}$ has both positive and negative terms. For this sequence, it happens that the sequence of absolute values, $\{|c_n|\}$, also converges to 0. You can show this by the Squeeze Theorem using the inequality

$$0 \le \frac{1}{n!} \le \frac{1}{2^n}, \quad n \ge 4.$$

In such cases, it is often convenient to consider the sequence of absolute values—and then apply Theorem 9.4, which states that if the absolute value sequence converges to 0, then the original signed sequence also converges to 0.

THEOREM 9.4 Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.$$

Proof Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$. Because both of these sequences converge to 0 and

$$-|a_n| \le a_n \le |a_n|$$

you can use the Squeeze Theorem to conclude that $\{a_n\}$ converges to 0.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Pattern Recognition for Sequences

Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the nth term of the sequence. In such cases, you may be required to discover a *pattern* in the sequence and to describe the nth term. Once the nth term has been specified, you can investigate the convergence or divergence of the sequence.

EXAMPLE 6 Finding the *n*th Term of a Sequence

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine whether the sequence you have chosen converges or diverges.

Solution First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers. By comparing a_n with n, you have the following pattern.

$$\frac{2^1}{1}, \frac{2^2}{3}, \frac{2^3}{5}, \frac{2^4}{7}, \frac{2^5}{9}, \ldots, \frac{2^n}{2n-1}, \ldots$$

Consider the function of a real variable $f(x) = 2^x/(2x - 1)$. Applying L'Hôpital's Rule produces

$$\lim_{x\to\infty} \frac{2^x}{2x-1} = \lim_{x\to\infty} \frac{2^x(\ln 2)}{2} = \infty.$$

Next, apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{2^n}{2n-1} = \infty.$$

So, the sequence diverges.

Without a specific rule for generating the terms of a sequence or some knowledge of the context in which the terms of the sequence are obtained, it is not possible to determine the convergence or divergence of the sequence merely from its first several terms. For instance, although the first three terms of the following four sequences are identical, the first two sequences converge to 0, the third sequence converges to $\frac{1}{9}$, and the fourth sequence diverges.

$$\{a_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\{b_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

$$\{c_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{7}{62}, \ldots, \frac{n^2-3n+3}{9n^2-25n+18}, \ldots$$

$$\{d_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, \ldots, \frac{-n(n+1)(n-4)}{6(n^2+3n-2)}, \ldots$$

The process of determining an nth term from the pattern observed in the first several terms of a sequence is an example of *inductive reasoning*.

EXAMPLE 7 Finding the nth Term of a Sequence

Determine the nth term for a sequence whose first five terms are

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \ldots$$

and then decide whether the sequence converges or diverges.

Solution Note that the numerators are 1 less than 3^n.

$$3^1 - 1 = 2 \qquad 3^2 - 1 = 8 \qquad 3^3 - 1 = 26 \qquad 3^4 - 1 = 80 \qquad 3^5 - 1 = 242$$

So, you can reason that the numerators are given by the rule

$$3^n - 1.$$

Factoring the denominators produces

$$\begin{aligned} 1 &= 1 \\ 2 &= 1 \cdot 2 \\ 6 &= 1 \cdot 2 \cdot 3 \\ 24 &= 1 \cdot 2 \cdot 3 \cdot 4 \end{aligned}$$

and

$$120 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5.$$

This suggests that the denominators are represented by $n!$. Finally, because the signs alternate, you can write the nth term as

$$a_n = (-1)^n\left(\frac{3^n - 1}{n!}\right).$$

From the discussion about the growth of $n!$, it follows that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3^n - 1}{n!} = 0.$$

Applying Theorem 9.4, you can conclude that

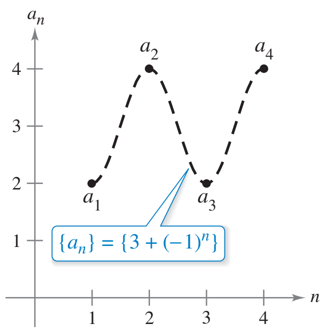
$$\lim_{n\to\infty} a_n = 0.$$

So, the sequence $\{a_n\}$ converges to 0.

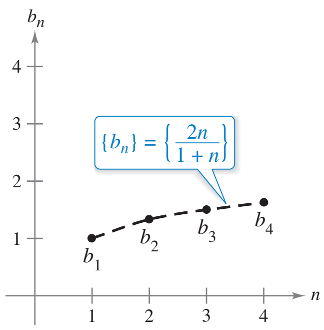
Monotonic Sequences and Bounded Sequences

So far, you have determined the convergence of a sequence by finding its limit. Even when you cannot determine the limit of a particular sequence, it still may be useful to know whether the sequence converges. Theorem 9.5 (on the next page) provides a test for convergence of sequences without determining the limit. First, some preliminary definitions are given.

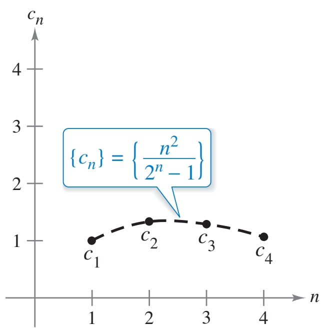
> **Definition of Monotonic Sequence**
>
> A sequence $\{a_n\}$ is **monotonic** when its terms are nondecreasing
>
> $$a_1 \le a_2 \le a_3 \le \cdot \cdot \cdot \le a_n \le \cdot \cdot \cdot$$
>
> or when its terms are nonincreasing
>
> $$a_1 \ge a_2 \ge a_3 \ge \cdot \cdot \cdot \ge a_n \ge \cdot \cdot \cdot .$$



(a) Not monotonic



(b) Monotonic



(c) Not monotonic

Figure 9.3

EXAMPLE 8 Determining Whether a Sequence Is Monotonic

Determine whether each sequence having the given nth term is monotonic.

a. $a_n = 3 + (-1)^n$

b. $b_n = \dfrac{2n}{1 + n}$

c. $c_n = \dfrac{n^2}{2^n - 1}$

Solution

a. This sequence alternates between 2 and 4. So, it is not monotonic.

b. This sequence is monotonic because each successive term is greater than its predecessor. To see this, compare the terms b_n and b_{n+1}. [Note that, because n is positive, you can multiply each side of the inequality by $(1 + n)$ and $(2 + n)$ without reversing the inequality sign.]

$$\begin{aligned} b_n = \frac{2n}{1+n} &\overset{?}{<} \frac{2(n+1)}{1+(n+1)} = b_{n+1} \\ 2n(2+n) &\overset{?}{<} (1+n)(2n+2) \\ 4n + 2n^2 &\overset{?}{<} 2 + 4n + 2n^2 \\ 0 &< 2 \end{aligned}$$

Starting with the final inequality, which is valid, you can reverse the steps to conclude that the original inequality is also valid.

c. This sequence is not monotonic, because the second term is greater than the first term, and greater than the third. (Note that when you drop the first term, the remaining sequence $c_2, c_3, c_4, . . .$ is monotonic.)

Figure 9.3 graphically illustrates these three sequences.

In Example 8(b), another way to see that the sequence is monotonic is to argue that the derivative of the corresponding differentiable function

$$f(x) = \frac{2x}{1 + x}$$

is positive for all x. This implies that f is increasing, which in turn implies that $\{b_n\}$ is increasing.

Definition of Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** when there is a real number M such that $a_n \le M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** when there is a real number N such that $N \le a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** when it is bounded above and bounded below.

Note that all three sequences in Example 3 (and shown in Figure 9.3) are bounded. To see this, note that

$$2 \le a_n \le 4, \quad 1 \le b_n \le 2, \quad \text{and} \quad 0 \le c_n \le \frac{4}{3}.$$

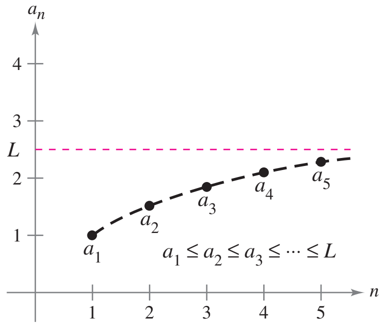
One important property of the real numbers is that they are **complete.** Informally, this means that there are no holes or gaps on the real number line. (The set of rational numbers does not have the completeness property.) The completeness axiom for real numbers can be used to conclude that if a sequence has an upper bound, then it must have a **least upper bound** (an upper bound that is less than all other upper bounds for the sequence). For example, the least upper bound of the sequence $\{a_n\} = \{n/(n+1)\}$,

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

is 1. The completeness axiom is used in the proof of Theorem 9.5.

THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.



Every bounded, nondecreasing sequence converges.
Figure 9.4

Proof Assume that the sequence is nondecreasing, as shown in Figure 9.4. For the sake of simplicity, also assume that each term in the sequence is positive. Because the sequence is bounded, there must exist an upper bound M such that

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots \le M.$$

From the completeness axiom, it follows that there is a least upper bound L such that

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots \le L.$$

For $\varepsilon > 0$, it follows that $L - \varepsilon < L$, and therefore $L - \varepsilon$ cannot be an upper bound for the sequence. Consequently, at least one term of $\{a_n\}$ is greater than $L - \varepsilon$. That is, $L - \varepsilon < a_N$ for some positive integer N. Because the terms of $\{a_n\}$ are nondecreasing, it follows that $a_N \le a_n$ for $n > N$. You now know that $L - \varepsilon < a_N \le a_n \le L < L + \varepsilon$, for every $n > N$. It follows that $|a_n - L| < \varepsilon$ for $n > N$, which by definition means that $\{a_n\}$ converges to L. The proof for a nonincreasing sequence is similar (see Exercise 91).

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 9 Bounded and Monotonic Sequences

a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic, and so, by Theorem 9.5, it must converge.

b. The divergent sequence $\{b_n\} = \{n^2/(n+1)\}$ is monotonic, but not bounded. (It is bounded below.)

c. The divergent sequence $\{c_n\} = \{(-1)^n\}$ is bounded, but not monotonic.

9.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Listing the Terms of a Sequence In Exercises 1–6, write the first five terms of the sequence.

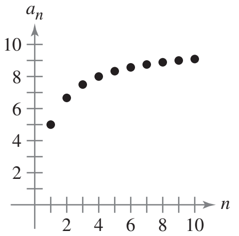
1. $a_n = 3^n$
2. $a_n = \left(-\frac{2}{5}\right)^n$
3. $a_n = \sin \frac{n\pi}{2}$
4. $a_n = \frac{3n}{n+4}$
5. $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$
6. $a_n = 2 + \frac{2}{n} - \frac{1}{n^2}$

Listing the Terms of a Sequence In Exercises 7 and 8, write the first five terms of the recursively defined sequence.

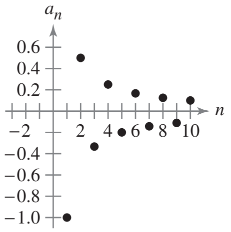
7. $a_1 = 3,\ a_{k+1} = 2(a_k - 1)$
8. $a_1 = 6,\ a_{k+1} = \frac{1}{3}a_k^2$

Matching In Exercises 9–12, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

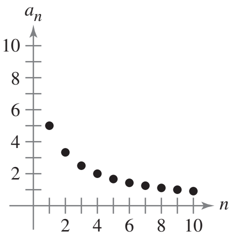
(a)



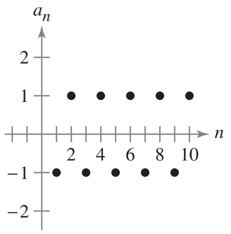
(b)



(c)



(d)



9. $a_n = \frac{10}{n+1}$
10. $a_n = \frac{10n}{n+1}$
11. $a_n = (-1)^n$
12. $a_n = \frac{(-1)^n}{n}$

Writing Terms In Exercises 13–16, write the next two apparent terms of the sequence. Describe the pattern you used to find these terms.

13. 2, 5, 8, 11, . . .
14. 8, 13, 18, 23, 28, . . .
15. 5, 10, 20, 40, . . .
16. $6, -2, \frac{2}{3}, -\frac{2}{9}, \ldots$

Simplifying Factorials In Exercises 17–20, simplify the ratio of factorials.

17. $\frac{(n+1)!}{n!}$
18. $\frac{n!}{(n+2)!}$
19. $\frac{(2n-1)!}{(2n+1)!}$
20. $\frac{(2n+2)!}{(2n)!}$

Finding the Limit of a Sequence In Exercises 21–24, find the limit (if possible) of the sequence.

21. $a_n = \frac{5n^2}{n^2+2}$
22. $a_n = 6 + \frac{2}{n^2}$
23. $a_n = \frac{2n}{\sqrt{n^2+1}}$
24. $a_n = \cos \frac{2}{n}$

Finding the Limit of a Sequence In Exercises 25–28, use a graphing utility to graph the first 10 terms of the sequence. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.

25. $a_n = \frac{4n+1}{n}$
26. $a_n = \frac{1}{n^{3/2}}$
27. $a_n = \sin \frac{n\pi}{2}$
28. $a_n = 2 - \frac{1}{4^n}$

Determining Convergence or Divergence In Exercises 29–44, determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

29. $a_n = \frac{5}{n+2}$
30. $a_n = 8 + \frac{5}{n}$
31. $a_n = (-1)^n\left(\frac{n}{n+1}\right)$
32. $a_n = \frac{1 + (-1)^n}{n^2}$
33. $a_n = \frac{10n^2 + 3n + 7}{2n^2 - 6}$
34. $a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$
35. $a_n = \frac{\ln(n^3)}{2n}$
36. $a_n = \frac{5^n}{3^n}$
37. $a_n = \frac{(n+1)!}{n!}$
38. $a_n = \frac{(n-2)!}{n!}$
39. $a_n = \frac{n^p}{e^n},\ p > 0$
40. $a_n = n \sin \frac{1}{n}$
41. $a_n = 2^{1/n}$
42. $a_n = -3^{-n}$
43. $a_n = \frac{\sin n}{n}$
44. $a_n = \frac{\cos \pi n}{n^2}$

Finding the nth Term of a Sequence In Exercises 45–52, write an expression for the nth term of the sequence. (There is more than one correct answer.)

45. 2, 8, 14, 20, . . .
46. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$
47. −2, 1, 6, 13, 22, . . .
48. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \ldots$
49. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$
50. 2, 24, 720, 40,320, 3,628,800, . . .
51. $2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$
52. $\frac{1}{2 \cdot 3}, \frac{2}{3 \cdot 4}, \frac{3}{4 \cdot 5}, \frac{4}{5 \cdot 6}, \ldots$

Finding Monotonic and Bounded Sequences In Exercises 53–60, determine whether the sequence with the given nth term is monotonic and whether it is bounded. Use a graphing utility to confirm your results.

53. $a_n = 4 - \dfrac{1}{n}$

54. $a_n = \dfrac{3n}{n+2}$

55. $a_n = ne^{-n/2}$

56. $a_n = \left(-\dfrac{2}{3}\right)^n$

57. $a_n = \left(\dfrac{2}{3}\right)^n$

58. $a_n = \left(\dfrac{3}{2}\right)^n$

59. $a_n = \sin \dfrac{n\pi}{6}$

60. $a_n = \dfrac{\cos n}{n}$

Using a Theorem In Exercises 61–64, (a) use Theorem 9.5 to show that the sequence with the given nth term converges, and (b) use a graphing utility to graph the first 10 terms of the sequence and find its limit.

61. $a_n = 7 + \dfrac{1}{n}$

62. $a_n = 5 - \dfrac{2}{n}$

63. $a_n = \dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)$

64. $a_n = 2 + \dfrac{1}{5^n}$

65. Increasing Sequence Let $\{a_n\}$ be an increasing sequence such that $2 \le a_n \le 4$. Explain why $\{a_n\}$ has a limit. What can you conclude about the limit?

66. Monotonic Sequence Let $\{a_n\}$ be a monotonic sequence such that $a_n \le 1$. Discuss the convergence of $\{a_n\}$. When $\{a_n\}$ converges, what can you conclude about its limit?

67. Compound Interest

Consider the sequence $\{A_n\}$ whose nth term is given by

$$A_n = P\left(1 + \frac{r}{12}\right)^n$$

where P is the principal, A_n is the account balance after n months, and r is the interest rate compounded annually.

(a) Is $\{A_n\}$ a convergent sequence? Explain.

(b) Find the first 10 terms of the sequence when $P = \$10{,}000$ and $r = 0.055$.

68. Compound Interest A deposit of \$100 is made in an account at the beginning of each month at an annual interest rate of 3% compounded monthly. The balance in the account after n months is $A_n = 100(401)(1.0025^n - 1)$.

(a) Compute the first six terms of the sequence $\{A_n\}$.

(b) Find the balance in the account after 5 years by computing the 60th term of the sequence.

(c) Find the balance in the account after 20 years by computing the 240th term of the sequence.

WRITING ABOUT CONCEPTS

69. Sequence Is it possible for a sequence to converge to two different numbers? If so, give an example. If not, explain why not.

70. Defining Terms In your own words, define each of the following.

(a) Sequence (b) Convergence of a sequence

(c) Monotonic sequence (d) Bounded sequence

71. Writing a Sequence Give an example of a sequence satisfying the condition or explain why no such sequence exists. (Examples are not unique.)

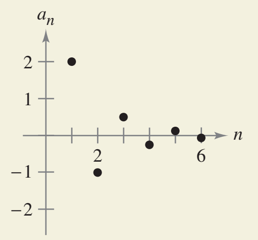
(a) A monotonically increasing sequence that converges to 10

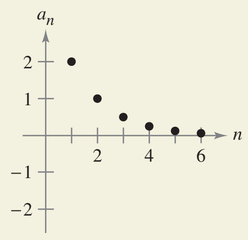
(b) A monotonically increasing bounded sequence that does not converge

(c) A sequence that converges to $\frac{3}{4}$

(d) An unbounded sequence that converges to 100

72. HOW DO YOU SEE IT? The graphs of two sequences are shown in the figures. Which graph represents the sequence with alternating signs? Explain.





73. Government Expenditures A government program that currently costs taxpayers \$4.5 billion per year is cut back by 20 percent per year.

(a) Write an expression for the amount budgeted for this program after n years.

(b) Compute the budgets for the first 4 years.

(c) Determine the convergence or divergence of the sequence of reduced budgets. If the sequence converges, find its limit.

74. Inflation When the rate of inflation is $4\frac{1}{2}\%$ per year and the average price of a car is currently \$25,000, the average price after n years is $P_n = \$25{,}000(1.045)^n$. Compute the average prices for the next 5 years.

75. Using a Sequence Compute the first six terms of the sequence $\{a_n\} = \left\{\sqrt[n]{n}\right\}$. If the sequence converges, find its limit.

76. Using a Sequence Compute the first six terms of the sequence

$$\{a_n\} = \left\{\left(1 + \frac{1}{n}\right)^n\right\}.$$

If the sequence converges, find its limit.

Lisa S./Shutterstock.com

77. Proof Prove that if $\{s_n\}$ converges to L and $L > 0$, then there exists a number N such that $s_n > 0$ for $n > N$.

78. Modeling Data The amounts of the federal debt a_n (in trillions of dollars) of the United States from 2000 through 2011 are given below as ordered pairs of the form (n, a_n), where n represents the year, with $n = 0$ corresponding to 2000. *(Source: U.S. Office of Management and Budget)*

(0, 5.6), (1, 5.8), (2, 6.2), (3, 6.8), (4, 7.4), (5, 7.9), (6, 8.5), (7, 9.0), (8, 10.0), (9, 11.9), (10, 13.5), (11, 14.8)

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = bn^2 + cn + d, \quad n = 0, 1, \ldots, 11$$

for the data. Use the graphing utility to plot the points and graph the model.

(b) Use the model to predict the amount of the federal debt in the year 2020.

True or False? **In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

79. If $\{a_n\}$ converges to 3 and $\{b_n\}$ converges to 2, then $\{a_n + b_n\}$ converges to 5.

80. If $\{a_n\}$ converges, then $\lim_{n\to\infty} (a_n - a_{n+1}) = 0$.

81. If $\{a_n\}$ converges, then $\{a_n/n\}$ converges to 0.

82. If $\{a_n\}$ diverges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

83. Fibonacci Sequence In a study of the progeny of rabbits, Fibonacci (ca. 1170–ca. 1240) encountered the sequence now bearing his name. The sequence is defined recursively as $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Write the first 12 terms of the sequence.

(b) Write the first 10 terms of the sequence defined by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

(c) Using the definition in part (b), show that

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

(d) The **golden ratio** ρ can be defined by $\lim_{n\to\infty} b_n = \rho$. Show that

$$\rho = 1 + \frac{1}{\rho}$$

and solve this equation for ρ.

84. Using a Theorem Show that the converse of Theorem 9.1 is not true. [*Hint:* Find a function $f(x)$ such that $f(n) = a_n$ converges, but $\lim_{x\to\infty} f(x)$ does not exist.]

85. Using a Sequence Consider the sequence

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

(a) Compute the first five terms of this sequence.

(b) Write a recursion formula for a_n, for $n \geq 2$.

(c) Find $\lim_{n\to\infty} a_n$.

86. Using a Sequence Consider the sequence $\{a_n\}$ where $a_1 = \sqrt{k}$, $a_{n+1} = \sqrt{k + a_n}$, and $k > 0$.

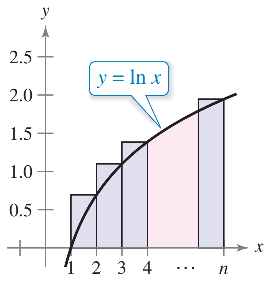
(a) Show that $\{a_n\}$ is increasing and bounded.

(b) Prove that $\lim_{n\to\infty} a_n$ exists.

(c) Find $\lim_{n\to\infty} a_n$.

87. Squeeze Theorem

(a) Show that $\int_1^n \ln x \, dx < \ln(n!)$ for $n \geq 2$.



(b) Draw a graph similar to the one above that shows

$$\ln(n!) < \int_1^{n+1} \ln x \, dx.$$

(c) Use the results of parts (a) and (b) to show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}, \text{ for } n > 1.$$

(d) Use the Squeeze Theorem for Sequences and the result of part (c) to show that $\lim_{n\to\infty} \left(\sqrt[n]{n!}/n\right) = 1/e$.

(e) Test the result of part (d) for $n = 20$, 50, and 100.

88. Proof Prove, using the definition of the limit of a sequence, that

$$\lim_{n\to\infty} \frac{1}{n^3} = 0.$$

89. Proof Prove, using the definition of the limit of a sequence, that $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$.

90. Using a Sequence Find a divergent sequence $\{a_n\}$ such that $\{a_{2n}\}$ converges.

91. Proof Prove Theorem 9.5 for a nonincreasing sequence.

PUTNAM EXAM CHALLENGE

92. Let $\{x_n\}$, $n \geq 0$, be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

93. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,

$$T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}.$$

The first few terms are

2, 3, 6, 14, 40, 152, 784, 5168, 40,576

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

9.2 Series and Convergence

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the *n*th-Term Test for Divergence of an infinite series.

Infinite Series

One important application of infinite sequences is in representing "infinite summations." Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot$$

Infinite Series

REMARK As you study this chapter, it is important to distinguish between an infinite series and a sequence. A sequence is an ordered collection of numbers

$a_1, a_2, a_3, \ldots, a_n, \ldots$

whereas a series is an infinite sum of terms from a sequence

$a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot .$

is an **infinite series** (or simply a **series**). The numbers a_1, a_2, a_3, and so on are the **terms** of the series. For some series, it is convenient to begin the index at $n = 0$ (or some other integer). As a typesetting convention, it is common to represent an infinite series as $\Sigma\, a_n$. In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the **sequence of partial sums** listed below.

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 \\
S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n
\end{aligned}$$

If this sequence of partial sums converges, then the series is said to converge and has the sum indicated in the next definition.

INFINITE SERIES

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double the intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdot \cdot \cdot + \frac{n}{2^n} + \cdot \cdot \cdot$$

is 2.

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is

$$S_n = a_1 + a_2 + \cdot \cdot \cdot + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series **diverges.**

As you study this chapter, you will see that there are two basic questions involving infinite series.

- Does a series converge or does it diverge?
- When a series converges, what is its sum?

These questions are not always easy to answer, especially the second one.

TECHNOLOGY Figure 9.5 shows the first 15 partial sums of the infinite series in Example 1(a). Notice how the values appear to approach the line $y = 1$.

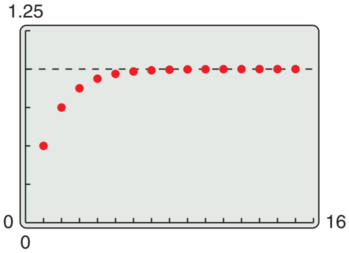


Figure 9.5

EXAMPLE 1 Convergent and Divergent Series

a. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdot \cdot \cdot$$

has the partial sums listed below. (You can also determine the partial sums of the series geometrically, as shown in Figure 9.6.)

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdot \cdot \cdot + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

Because

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$$

it follows that the series converges and its sum is 1.

b. The nth partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdot \cdot \cdot$$

is

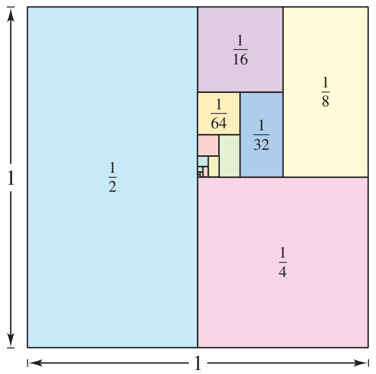
$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of S_n is 1, the series converges and its sum is 1.

c. The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdot \cdot \cdot$$

diverges because $S_n = n$ and the sequence of partial sums diverges.



You can determine the partial sums of the series in Example 1(a) geometrically using this figure.
Figure 9.6

The series in Example 1(b) is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdot \cdot \cdot . \qquad \text{Telescoping series}$$

Note that b_2 is canceled by the second term, b_3 is canceled by the third term, and so on. Because the nth partial sum of this series is

$$S_n = b_1 - b_{n+1}$$

it follows that a telescoping series will converge if and only if b_n approaches a finite number as $n \to \infty$. Moreover, if the series converges, then its sum is

$$S = b_1 - \lim_{n\to\infty} b_{n+1}.$$

FOR FURTHER INFORMATION To learn more about the partial sums of infinite series, see the article "Six Ways to Sum a Series" by Dan Kalman in *The College Mathematics Journal.* To view this article, go to *MathArticles.com.*

EXAMPLE 2 **Writing a Series in Telescoping Form**

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Solution

Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}.$$

From this telescoping form, you can see that the nth partial sum is

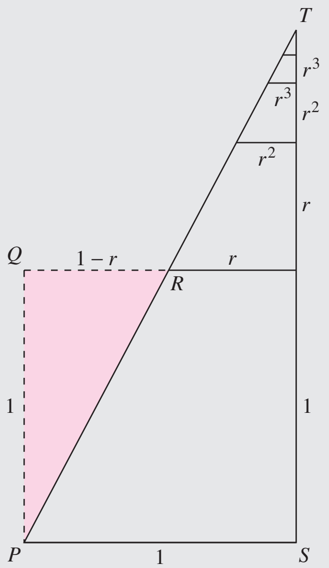
$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdot \cdot \cdot + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2n+1}\right) = 1.$$

Exploration

In "Proof Without Words," by Benjamin G. Klein and Irl C. Bivens, the authors present the diagram below. Explain why the second statement after the diagram is valid. How is this result related to Theorem 9.6?



$\Delta PQR \sim \Delta TSP$

$$1 + r + r^2 + r^3 + \cdot \cdot \cdot = \frac{1}{1-r}$$

Exercise taken from "Proof Without Words" by Benjamin G. Klein and Irl C. Bivens, *Mathematics Magazine,* 61, No. 4, October 1988, p. 219, by permission of the authors.

Geometric Series

The series in Example 1(a) is a **geometric series.** In general, the series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdot \cdot \cdot + ar^n + \cdot \cdot \cdot, \quad a \neq 0$$

Geometric series

is a **geometric series** with ratio r, $r \neq 0$.

THEOREM 9.6 **Convergence of a Geometric Series**

A geometric series with ratio r diverges when $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Proof It is easy to see that the series diverges when $r = \pm 1$. If $r \neq \pm 1$, then

$$S_n = a + ar + ar^2 + \cdot \cdot \cdot + ar^{n-1}.$$

Multiplication by r yields

$$rS_n = ar + ar^2 + ar^3 + \cdot \cdot \cdot + ar^n.$$

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^n$. Therefore, $S_n(1 - r) = a(1 - r^n)$, and the nth partial sum is

$$S_n = \frac{a}{1-r}(1 - r^n).$$

When $0 < |r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{a}{1-r}(1 - r^n)\right] = \frac{a}{1-r}\left[\lim_{n\to\infty} (1 - r^n)\right] = \frac{a}{1-r}$$

which means that the series *converges* and its sum is $a/(1 - r)$. It is left to you to show that the series diverges when $|r| > 1$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

TECHNOLOGY Try using a graphing utility to compute the sum of the first 20 terms of the sequence in Example 3(a). You should obtain a sum of about 5.999994.

EXAMPLE 3 **Convergent and Divergent Geometric Series**

a. The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$. Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1-r} = \frac{3}{1-(1/2)} = 6.$$

b. The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

has a ratio of $r = \frac{3}{2}$. Because $|r| \geq 1$, the series diverges.

The formula for the sum of a geometric series can be used to write a repeating decimal as the ratio of two integers, as demonstrated in the next example.

EXAMPLE 4 **A Geometric Series for a Repeating Decimal**

See LarsonCalculus.com for an interactive version of this type of example.

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Solution For the repeating decimal $0.\overline{08}$, you can write

$$0.080808\ldots = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \cdots$$
$$= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n.$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808\ldots = \frac{a}{1-r} = \frac{8/10^2}{1-(1/10^2)} = \frac{8}{99}.$$

Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$.

The convergence of a series is not affected by the removal of a finite number of terms from the beginning of the series. For instance, the geometric series

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

both converge. Furthermore, because the sum of the second series is

$$\frac{a}{1-r} = \frac{1}{1-(1/2)} = 2$$

you can conclude that the sum of the first series is

$$S = 2 - \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right]$$
$$= 2 - \frac{15}{8}$$
$$= \frac{1}{8}.$$

The properties in the next theorem are direct consequences of the corresponding properties of limits of sequences.

THEOREM 9.7 Properties of Infinite Series

Let $\Sigma\, a_n$ and $\Sigma\, b_n$ be convergent series, and let A, B, and c be real numbers. If $\Sigma\, a_n = A$ and $\Sigma\, b_n = B$, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$
2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

*n*th-Term Test for Divergence

The next theorem states that when a series converges, the limit of its nth term must be 0.

REMARK Be sure you see that the converse of Theorem 9.8 is generally not true. That is, if the sequence $\{a_n\}$ converges to 0, then the series $\Sigma\, a_n$ may either converge or diverge.

THEOREM 9.8 Limit of the *n*th Term of a Convergent Series

If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Proof Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$$

it follows that

$$\begin{aligned} L &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} (S_{n-1} + a_n) \\ &= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n \\ &= L + \lim_{n\to\infty} a_n \end{aligned}$$

which implies that $\{a_n\}$ converges to 0.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The contrapositive of Theorem 9.8 provides a useful test for *divergence*. This ***n*th-Term Test for Divergence** states that if the limit of the nth term of a series does *not* converge to 0, then the series must diverge.

THEOREM 9.9 *n*th-Term Test for Divergence

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE 5 Using the *n*th-Term Test for Divergence

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n\to\infty} 2^n = \infty.$$

So, the limit of the nth term is not 0, and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have

$$\lim_{n\to\infty} \frac{n!}{2n! + 1} = \frac{1}{2}.$$

So, the limit of the nth term is not 0, and the series diverges.

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Because the limit of the nth term is 0, the nth-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence. (In the next section, you will see that this particular series diverges.)

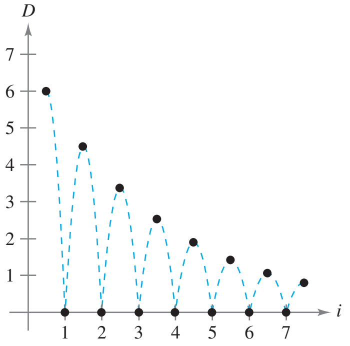
REMARK The series in Example 5(c) will play an important role in this chapter.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

You will see that this series diverges even though the nth term approaches 0 as n approaches ∞.

EXAMPLE 6 Bouncing Ball Problem

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



The height of each bounce is three-fourths the height of the preceding bounce.
Figure 9.7

Solution When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let D_i be the distance traveled up and down. For example, D_2 and D_3 are

$$D_2 = \underbrace{6\left(\frac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\frac{3}{4}\right)}_{\text{Down}} = 12\left(\frac{3}{4}\right)$$

and

$$D_3 = \underbrace{6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}_{\text{Down}} = 12\left(\frac{3}{4}\right)^2.$$

By continuing this process, it can be determined that the total vertical distance is

$$\begin{aligned} D &= 6 + 12\left(\frac{3}{4}\right) + 12\left(\frac{3}{4}\right)^2 + 12\left(\frac{3}{4}\right)^3 + \cdots \\ &= 6 + 12\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n+1} \\ &= 6 + 12\left(\frac{3}{4}\right)\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n} \\ &= 6 + 9\left[\frac{1}{1-(3/4)}\right] \\ &= 6 + 9(4) \\ &= 42 \text{ feet.} \end{aligned}$$

9.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Partial Sums In Exercises 1–6, find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5.

1. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

2. $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \frac{5}{6 \cdot 7} + \cdots$

3. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$

4. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$

5. $\sum_{n=1}^{\infty} \frac{3}{2^{n-1}}$ **6.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

Verifying Divergence In Exercises 7–14, verify that the infinite series diverges.

7. $\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$ **8.** $\sum_{n=0}^{\infty} 4(-1.05)^n$

9. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ **10.** $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

11. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$ **12.** $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

13. $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$ **14.** $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

Verifying Convergence In Exercises 15–20, verify that the infinite series converges.

15. $\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$ **16.** $\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$

17. $\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$

18. $\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$

19. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (*Hint*: Use partial fractions.)

20. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (*Hint*: Use partial fractions.)

Numerical, Graphical, and Analytic Analysis In Exercises 21–24, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	20	50	100
S_n					

21. $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$ **22.** $\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$

23. $\sum_{n=1}^{\infty} 2(0.9)^{n-1}$ **24.** $\sum_{n=1}^{\infty} 10\left(-\frac{1}{4}\right)^{n-1}$

Finding the Sum of a Convergent Series In Exercises 25–34, find the sum of the convergent series.

25. $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$ **26.** $\sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n$

27. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$ **28.** $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$

29. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$ **30.** $9 - 3 + 1 - \frac{1}{3} + \cdots$

31. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$ **32.** $\sum_{n=0}^{\infty} [(0.3)^n + (0.8)^n]$

33. $\sum_{n=1}^{\infty} (\sin 1)^n$ **34.** $\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2}$

Using a Geometric Series In Exercises 35–40, (a) write the repeating decimal as a geometric series, and (b) write its sum as the ratio of two integers.

35. $0.\overline{4}$ **36.** $0.\overline{36}$

37. $0.\overline{81}$ **38.** $0.\overline{01}$

39. $0.0\overline{75}$ **40.** $0.2\overline{15}$

Determining Convergence or Divergence In Exercises 41–54, determine the convergence or divergence of the series.

41. $\sum_{n=0}^{\infty} (1.075)^n$ **42.** $\sum_{n=0}^{\infty} \frac{3^n}{1000}$

43. $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$ **44.** $\sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$

45. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$ **46.** $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

47. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ **48.** $\sum_{n=0}^{\infty} \frac{3}{5^n}$

49. $\sum_{n=2}^{\infty} \frac{n}{\ln n}$ **50.** $\sum_{n=1}^{\infty} \ln \frac{1}{n}$

51. $\sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$ **52.** $\sum_{n=1}^{\infty} e^{-n}$

53. $\sum_{n=1}^{\infty} \arctan n$ **54.** $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

WRITING ABOUT CONCEPTS

55. Series State the definitions of convergent and divergent series.

56. Sequence and Series Describe the difference between $\lim_{n\to\infty} a_n = 5$ and $\sum_{n=1}^{\infty} a_n = 5$.

WRITING ABOUT CONCEPTS (continued)

57. Geometric Series Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

58. *n*th-Term Test for Divergence State the nth-Term Test for Divergence.

59. Comparing Series Explain any differences among the following series.

(a) $\sum_{n=1}^{\infty} a_n$ (b) $\sum_{k=1}^{\infty} a_k$ (c) $\sum_{n=1}^{\infty} a_k$

60. Using a Series

(a) You delete a finite number of terms from a divergent series. Will the new series still diverge? Explain your reasoning.

(b) You add a finite number of terms to a convergent series. Will the new series still converge? Explain your reasoning.

Making a Series Converge **In Exercises 61–66, find all values of x for which the series converges. For these values of x, write the sum of the series as a function of x.**

61. $\sum_{n=1}^{\infty} (3x)^n$

62. $\sum_{n=0}^{\infty} \left(\frac{2}{x}\right)^n$

63. $\sum_{n=1}^{\infty} (x-1)^n$

64. $\sum_{n=0}^{\infty} 5\left(\frac{x-2}{3}\right)^n$

65. $\sum_{n=0}^{\infty} (-1)^n x^n$

66. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

Using a Geometric Series **In Exercises 67 and 68, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums S_3 and S_5. What do you notice?**

67. $1 + x + x^2 + x^3 + \cdots$

68. $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots$

Writing **In Exercises 69 and 70, use a graphing utility to determine the first term that is less than 0.0001 in each of the convergent series. Note that the answers are very different. Explain how this will affect the rate at which the series converges.**

69. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$

70. $\sum_{n=1}^{\infty} \frac{1}{2^n}, \quad \sum_{n=1}^{\infty} (0.01)^n$

71. Marketing An electronic games manufacturer producing a new product estimates the annual sales to be 8000 units. Each year, 5% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.95(8000)]$ units will be in use after 2 years, and so on. How many units will be in use after n years?

72. Depreciation A company buys a machine for \$475,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after n years. What is its value after 5 years?

73. Multiplier Effect

The total annual spending by tourists in a resort city is \$200 million. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the same city, and so on. Write the geometric series that gives the total amount of spending generated by the \$200 million and find the sum of the series.

74. Multiplier Effect Repeat Exercise 73 when the percent of the revenue that is spent again in the city decreases to 60%.

75. Distance A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet. Find the total distance traveled by the ball.

76. Time The ball in Exercise 75 takes the following times for each fall.

$$\begin{array}{ll}
s_1 = -16t^2 + 16, & s_1 = 0 \text{ when } t = 1 \\
s_2 = -16t^2 + 16(0.81), & s_2 = 0 \text{ when } t = 0.9 \\
s_3 = -16t^2 + 16(0.81)^2, & s_3 = 0 \text{ when } t = (0.9)^2 \\
s_4 = -16t^2 + 16(0.81)^3, & s_4 = 0 \text{ when } t = (0.9)^3 \\
\vdots & \vdots \\
s_n = -16t^2 + 16(0.81)^{n-1}, & s_n = 0 \text{ when } t = (0.9)^{n-1}
\end{array}$$

Beginning with s_2, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by

$$t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n.$$

Find this total time.

Probability **In Exercises 77 and 78, the random variable n represents the number of units of a product sold per day in a store. The probability distribution of n is given by $P(n)$. Find the probability that two units are sold in a given day $[P(2)]$ and show that $P(0) + P(1) + P(2) + P(3) + \cdots = 1$.**

77. $P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$

78. $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

79. Probability A fair coin is tossed repeatedly. The probability that the first head occurs on the nth toss is given by $P(n) = \left(\frac{1}{2}\right)^n$, where $n \geq 1$.

(a) Show that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$.

(b) The expected number of tosses required until the first head occurs in the experiment is given by

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n.$$

Is this series geometric?

(c) Use a computer algebra system to find the sum in part (b).

AISPIX by Image Source/Shutterstock.com

80. Probability In an experiment, three people toss a fair coin one at a time until one of them tosses a head. Determine, for each person, the probability that he or she tosses the first head. Verify that the sum of the three probabilities is 1.

81. Area The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded (see figure). Determine the area of the shaded regions (a) when this process is continued five more times, and (b) when this pattern of shading is continued infinitely.

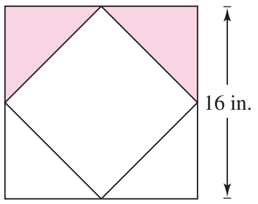


Figure for 81

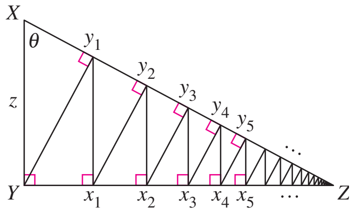


Figure for 82

82. Length A right triangle XYZ is shown above where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle, as shown in the figure.

(a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of z and θ.

(b) Find the total length of the perpendicular line segments when $z = 1$ and $\theta = \pi/6$.

Using a Geometric Series **In Exercises 83–86, use the formula for the nth partial sum of a geometric series**

$$\sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}.$$

83. Present Value The winner of a $2,000,000 sweepstakes will be paid $100,000 per year for 20 years. The money earns 6% interest per year. The present value of the winnings is $\sum_{n=1}^{20} 100{,}000\left(\frac{1}{1.06}\right)^n$. Compute the present value and interpret its meaning.

84. Annuities When an employee receives a paycheck at the end of each month, P dollars is invested in a retirement account. These deposits are made each month for t years and the account earns interest at the annual percentage rate r. When the interest is compounded monthly, the amount A in the account at the end of t years is

$$A = P + P\left(1 + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{12t-1}$$
$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right].$$

When the interest is compounded continuously, the amount A in the account after t years is

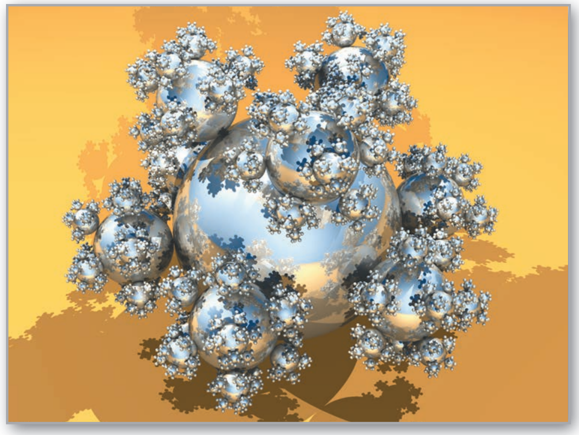
$$A = P + Pe^{r/12} + Pe^{2r/12} + Pe^{(12t-1)r/12}$$
$$= \frac{P(e^{rt} - 1)}{e^{r/12} - 1}.$$

Verify the formulas for the sums given above.

85. Salary You go to work at a company that pays $0.01 for the first day, $0.02 for the second day, $0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?

86. Sphereflake

The sphereflake shown below is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely. Prove that the sphereflake has an infinite surface area.



Annuities **In Exercises 87–90, consider making monthly deposits of P dollars in a savings account at an annual interest rate r. Use the results of Exercise 84 to find the balance A after t years when the interest is compounded (a) monthly and (b) continuously.**

87. $P = \$45$, $r = 3\%$, $t = 20$ years

88. $P = \$75$, $r = 5.5\%$, $t = 25$ years

89. $P = \$100$, $r = 4\%$, $t = 35$ years

90. $P = \$30$, $r = 6\%$, $t = 50$ years

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

92. If $\sum_{n=1}^{\infty} a_n = L$, then $\sum_{n=0}^{\infty} a_n = L + a_0$.

93. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$.

94. The series $\sum_{n=1}^{\infty} \frac{n}{1000(n+1)}$ diverges.

95. $0.75 = 0.749999\ldots$

96. Every decimal with a repeating pattern of digits is a rational number.

Courtesy of Eric Haines

97. Using Divergent Series Find two divergent series Σa_n and Σb_n such that $\Sigma(a_n + b_n)$ converges.

98. Proof Given two infinite series Σa_n and Σb_n such that Σa_n converges and Σb_n diverges, prove that $\Sigma(a_n + b_n)$ diverges.

99. Fibonacci Sequence The Fibonacci sequence is defined recursively by $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Show that $\dfrac{1}{a_{n+1}\, a_{n+3}} = \dfrac{1}{a_{n+1}\, a_{n+2}} - \dfrac{1}{a_{n+2}\, a_{n+3}}$.

(b) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a_{n+1}\, a_{n+3}} = 1$.

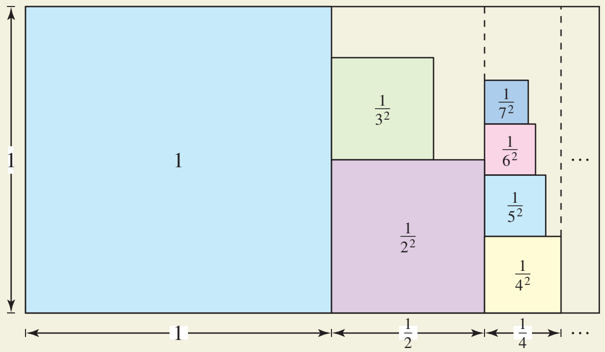
100. Remainder Let Σa_n be a convergent series, and let

$$R_N = a_{N+1} + a_{N+2} + \cdots$$

be the remainder of the series after the first N terms. Prove that $\displaystyle\lim_{N\to\infty} R_N = 0$.

101. Proof Prove that $\dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \cdots = \dfrac{1}{r-1}$, for $|r| > 1$.

102. HOW DO YOU SEE IT? The figure below represents an informal way of showing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$. Explain how the figure implies this conclusion.



FOR FURTHER INFORMATION For more on this exercise, see the article "Convergence with Pictures" by P. J. Rippon in *American Mathematical Monthly*.

PUTNAM EXAM CHALLENGE

103. Express $\displaystyle\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number.

104. Let $f(n)$ be the sum of the first n terms of the sequence 0, 1, 1, 2, 2, 3, 3, 4, . . . , where the nth term is given by

$$a_n = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}.$$

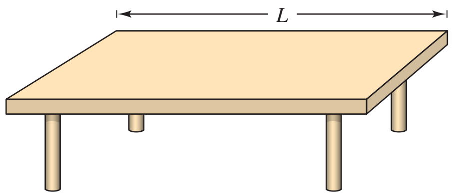
Show that if x and y are positive integers and $x > y$ then $xy = f(x + y) - f(x - y)$.

SECTION PROJECT

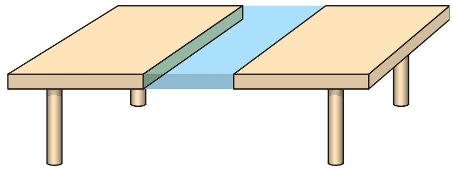
Cantor's Disappearing Table

The following procedure shows how to make a table disappear by removing only half of the table!

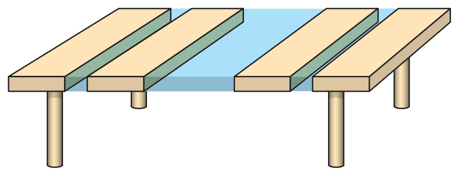
(a) Original table has a length of L.



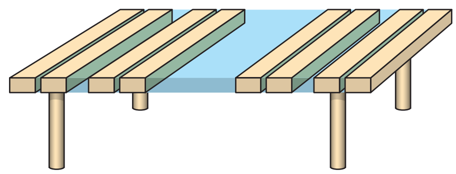
(b) Remove $\frac{1}{4}$ of the table centered at the midpoint. Each remaining piece has a length that is less than $\frac{1}{2}L$.



(c) Remove $\frac{1}{8}$ of the table by taking sections of length $\frac{1}{16}L$ from the centers of each of the two remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8}$ of the table. Each remaining piece has a length that is less than $\frac{1}{4}L$.



(d) Remove $\frac{1}{16}$ of the table by taking sections of length $\frac{1}{64}L$ from the centers of each of the four remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ of the table. Each remaining piece has a length that is less than $\frac{1}{8}L$.



Will continuing this process cause the table to disappear, even though you have only removed half of the table? Why?

FOR FURTHER INFORMATION Read the article "Cantor's Disappearing Table" by Larry E. Knop in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

9.3 The Integral Test and p-Series

- **Use the Integral Test to determine whether an infinite series converges or diverges.**
- **Use properties of p-series and harmonic series.**

The Integral Test

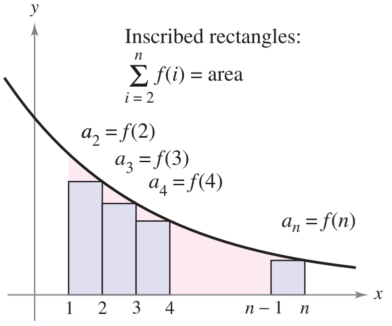
In this and the next section, you will study several convergence tests that apply to series with *positive* terms.

THEOREM 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.



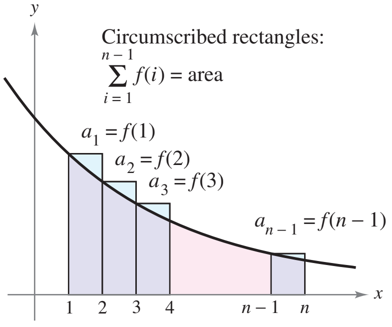


Figure 9.8

Proof Begin by partitioning the interval $[1, n]$ into $(n - 1)$ unit intervals, as shown in Figure 9.8. The total areas of the inscribed rectangles and the circumscribed rectangles are

$$\sum_{i=2}^{n} f(i) = f(2) + f(3) + \cdots + f(n) \qquad \text{Inscribed area}$$

and

$$\sum_{i=1}^{n-1} f(i) = f(1) + f(2) + \cdots + f(n-1). \qquad \text{Circumscribed area}$$

The exact area under the graph of f from $x = 1$ to $x = n$ lies between the inscribed and circumscribed areas.

$$\sum_{i=2}^{n} f(i) \leq \int_1^n f(x)\,dx \leq \sum_{i=1}^{n-1} f(i)$$

Using the nth partial sum, $S_n = f(1) + f(2) + \cdots + f(n)$, you can write this inequality as

$$S_n - f(1) \leq \int_1^n f(x)\,dx \leq S_{n-1}.$$

Now, assuming that $\int_1^{\infty} f(x)\,dx$ converges to L, it follows that for $n \geq 1$

$$S_n - f(1) \leq L \quad \Longrightarrow \quad S_n \leq L + f(1).$$

Consequently, $\{S_n\}$ is bounded and monotonic, and by Theorem 9.5 it converges. So, $\sum a_n$ converges. For the other direction of the proof, assume that the improper integral diverges. Then $\int_1^n f(x)\,dx$ approaches infinity as $n \to \infty$, and the inequality $S_{n-1} \geq \int_1^n f(x)\,dx$ implies that $\{S_n\}$ diverges. So, $\sum a_n$ diverges.

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

Remember that the convergence or divergence of $\sum a_n$ is not affected by deleting the first N terms. Similarly, when the conditions for the Integral Test are satisfied for all $x \geq N > 1$, you can simply use the integral $\int_N^{\infty} f(x)\,dx$ to test for convergence or divergence. (This is illustrated in Example 4.)

EXAMPLE 1 Using the Integral Test

Apply the Integral Test to the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Solution The function $f(x) = x/(x^2+1)$ is positive and continuous for $x \geq 1$. To determine whether f is decreasing, find the derivative.

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

So, $f'(x) < 0$ for $x > 1$ and it follows that f satisfies the conditions for the Integral Test. You can integrate to obtain

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int_1^{\infty} \frac{2x}{x^2+1}\,dx \\ &= \frac{1}{2}\lim_{b\to\infty}\int_1^{b} \frac{2x}{x^2+1}\,dx \\ &= \frac{1}{2}\lim_{b\to\infty}\Big[\ln(x^2+1)\Big]_1^b \\ &= \frac{1}{2}\lim_{b\to\infty}\left[\ln(b^2+1) - \ln 2\right] \\ &= \infty.\end{aligned}$$

So, the series *diverges*.

EXAMPLE 2 Using the Integral Test

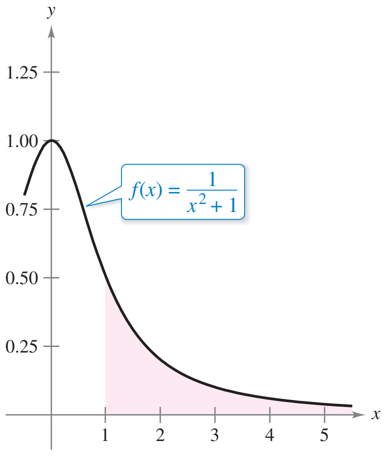
See LarsonCalculus.com for an interactive version of this type of example.

Apply the Integral Test to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution Because $f(x) = 1/(x^2+1)$ satisfies the conditions for the Integral Test (check this), you can integrate to obtain

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\,dx &= \lim_{b\to\infty}\int_1^{b} \frac{1}{x^2+1}\,dx \\ &= \lim_{b\to\infty}\Big[\arctan x\Big]_1^b \\ &= \lim_{b\to\infty}(\arctan b - \arctan 1) \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4}.\end{aligned}$$

So, the series *converges* (see Figure 9.9).



Because the improper integral converges, the infinite series also converges.
Figure 9.9

In Example 2, the fact that the improper integral converges to $\pi/4$ does not imply that the infinite series converges to $\pi/4$. To approximate the sum of the series, you can use the inequality

$$\sum_{n=1}^{N} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{N} \frac{1}{n^2+1} + \int_N^{\infty} \frac{1}{x^2+1}\,dx.$$

(See Exercise 54.) The larger the value of N, the better the approximation. For instance, using $N = 200$ produces $1.072 \leq \Sigma\, 1/(n^2+1) \leq 1.077$.

HARMONIC SERIES

Pythagoras and his students paid close attention to the development of music as an abstract science. This led to the discovery of the relationship between the tone and the length of a vibrating string. It was observed that the most beautiful musical harmonies corresponded to the simplest ratios of whole numbers. Later mathematicians developed this idea into the harmonic series, where the terms in the harmonic series correspond to the nodes on a vibrating string that produce multiples of the fundamental frequency. For example, $\frac{1}{2}$ is twice the fundamental frequency, $\frac{1}{3}$ is three times the fundamental frequency, and so on.

p-Series and Harmonic Series

In the remainder of this section, you will investigate a second type of series that has a simple arithmetic test for convergence or divergence. A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \qquad p\text{-series}$$

is a ***p*-series,** where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \qquad \text{Harmonic series}$$

is the **harmonic** series. A **general harmonic series** is of the form $\Sigma 1/(an + b)$. In music, strings of the same material, diameter, and tension, and whose lengths form a harmonic series, produce harmonic tones.

The Integral Test is convenient for establishing the convergence or divergence of p-series. This is shown in the proof of Theorem 9.11.

THEOREM 9.11 Convergence of *p*-Series

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

converges for $p > 1$, and diverges for $0 < p \le 1$.

Proof The proof follows from the Integral Test and from Theorem 8.5, which states that

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

converges for $p > 1$ and diverges for $0 < p \le 1$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 3 Convergent and Divergent *p*-Series

Discuss the convergence or divergence of (a) the harmonic series and (b) the p-series with $p = 2$.

Solution

a. From Theorem 9.11, it follows that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \qquad p = 1$$

diverges.

b. From Theorem 9.11, it follows that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \qquad p = 2$$

converges.

The sum of the series in Example 3(b) can be shown to be $\pi^2/6$. (This was proved by Leonhard Euler, but the proof is too difficult to present here.) Be sure you see that the Integral Test does not tell you that the sum of the series is equal to the value of the integral. For instance, the sum of the series in Example 3(b) is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645$$

whereas the value of the corresponding improper integral is

$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1.$$

EXAMPLE 4 Testing a Series for Convergence

Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

converges or diverges.

Solution This series is similar to the divergent harmonic series. If its terms were greater than those of the harmonic series, you would expect it to diverge. However, because its terms are less than those of the harmonic series, you are not sure what to expect. The function

$$f(x) = \frac{1}{x \ln x}$$

is positive and continuous for $x \geq 2$. To determine whether f is decreasing, first rewrite f as

$$f(x) = (x \ln x)^{-1}$$

and then find its derivative.

$$f'(x) = (-1)(x \ln x)^{-2}(1 + \ln x) = -\frac{1 + \ln x}{x^2(\ln x)^2}$$

So, $f'(x) < 0$ for $x > 2$ and it follows that f satisfies the conditions for the Integral Test.

$$\begin{aligned}\int_2^{\infty} \frac{1}{x \ln x}\,dx &= \int_2^{\infty} \frac{1/x}{\ln x}\,dx \\ &= \lim_{b\to\infty} \Big[\ln(\ln x)\Big]_2^b \\ &= \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] \\ &= \infty\end{aligned}$$

The series diverges.

Note that the infinite series in Example 4 diverges very slowly. For instance, as shown in the table, the sum of the first 10 terms is approximately 1.6878196, whereas the sum of the first 100 terms is just slightly greater: 2.3250871. In fact, the sum of the first 10,000 terms is approximately 3.0150217. You can see that although the infinite series "adds up to infinity," it does so very slowly.

n	11	101	1001	10,001	100,001
S_n	1.6878	2.3251	2.7275	3.0150	3.2382

9.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using the Integral Test In Exercises 1–22, confirm that the Integral Test can be applied to the series. Then use the Integral Test to determine the convergence or divergence of the series.

1. $\sum_{n=1}^{\infty} \frac{1}{n+3}$
2. $\sum_{n=1}^{\infty} \frac{2}{3n+5}$
3. $\sum_{n=1}^{\infty} \frac{1}{2^n}$
4. $\sum_{n=1}^{\infty} 3^{-n}$
5. $\sum_{n=1}^{\infty} e^{-n}$
6. $\sum_{n=1}^{\infty} ne^{-n/2}$
7. $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \cdots$
8. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$
9. $\frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} + \frac{\ln 5}{5} + \frac{\ln 6}{6} + \cdots$
10. $\frac{\ln 2}{\sqrt{2}} + \frac{\ln 3}{\sqrt{3}} + \frac{\ln 4}{\sqrt{4}} + \frac{\ln 5}{\sqrt{5}} + \frac{\ln 6}{\sqrt{6}} + \cdots$
11. $\frac{1}{\sqrt{1}(\sqrt{1}+1)} + \frac{1}{\sqrt{2}(\sqrt{2}+1)} + \frac{1}{\sqrt{3}(\sqrt{3}+1)} + \cdots + \frac{1}{\sqrt{n}(\sqrt{n}+1)} + \cdots$
12. $\frac{1}{4} + \frac{2}{7} + \frac{3}{12} + \cdots + \frac{n}{n^2+3} + \cdots$
13. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$
14. $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$
15. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$
16. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
17. $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$
18. $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$
19. $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$
20. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$
21. $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$
22. $\sum_{n=1}^{\infty} \frac{n}{n^4+2n^2+1}$

Using the Integral Test In Exercises 23 and 24, use the Integral Test to determine the convergence or divergence of the series, where k is a positive integer.

23. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+c}$
24. $\sum_{n=1}^{\infty} n^k e^{-n}$

Requirements of the Integral Test In Exercises 25–28, explain why the Integral Test does not apply to the series.

25. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
26. $\sum_{n=1}^{\infty} e^{-n} \cos n$
27. $\sum_{n=1}^{\infty} \frac{2+\sin n}{n}$
28. $\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$

Using the Integral Test In Exercises 29–32, use the Integral Test to determine the convergence or divergence of the p-series.

29. $\sum_{n=1}^{\infty} \frac{1}{n^3}$
30. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$
31. $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$
32. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

Using a p-Series In Exercises 33–38, use Theorem 9.11 to determine the convergence or divergence of the p-series.

33. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$
34. $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$
35. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$
36. $1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{25}} + \cdots$
37. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$
38. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

39. **Numerical and Graphical Analysis** Use a graphing utility to find the indicated partial sum S_n and complete the table. Then use a graphing utility to graph the first 10 terms of the sequence of partial sums. For each series, compare the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	20	50	100
S_n					

(a) $\sum_{n=1}^{\infty} 3\left(\frac{1}{5}\right)^{n-1} = \frac{15}{4}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

40. **Numerical Reasoning** Because the harmonic series diverges, it follows that for any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=1}^{N} \frac{1}{n} > M.$$

(a) Use a graphing utility to complete the table.

M	2	4	6	8
N				

(b) As the real number M increases in equal increments, does the number N increase in equal increments? Explain.

WRITING ABOUT CONCEPTS

41. Integral Test State the Integral Test and give an example of its use.

42. p-Series Define a p-series and state the requirements for its convergence.

43. Using a Series A friend in your calculus class tells you that the following series converges because the terms are very small and approach 0 rapidly. Is your friend correct? Explain.

$$\frac{1}{10{,}000} + \frac{1}{10{,}001} + \frac{1}{10{,}002} + \cdots$$

44. Using a Function Let f be a positive, continuous, and decreasing function for $x \geq 1$, such that $a_n = f(n)$. Use a graph to rank the following quantities in decreasing order. Explain your reasoning.

(a) $\displaystyle\sum_{n=2}^{7} a_n$ (b) $\displaystyle\int_1^7 f(x)\,dx$ (c) $\displaystyle\sum_{n=1}^{6} a_n$

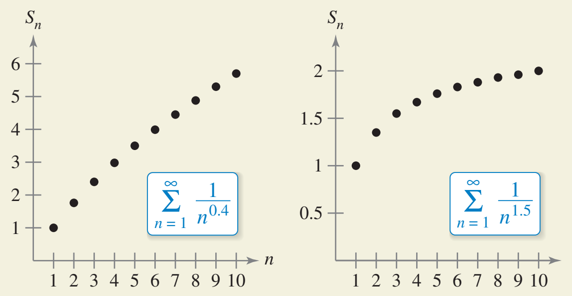
45. Using a Series Use a graph to show that the inequality is true. What can you conclude about the convergence or divergence of the series? Explain.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$

46. HOW DO YOU SEE IT? The graphs show the sequences of partial sums of the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.4}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1.5}}.$$

Using Theorem 9.11, the first series diverges and the second series converges. Explain how the graphs show this.



Finding Values **In Exercises 47–52, find the positive values of p for which the series converges.**

47. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

48. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$

50. $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

51. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{p}\right)^n$

52. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n[\ln(\ln n)]^p}$

53. Proof Let f be a positive, continuous, and decreasing function for $x \geq 1$, such that $a_n = f(n)$. Prove that if the series

$$\sum_{n=1}^{\infty} a_n$$

converges to S, then the remainder $R_N = S - S_N$ is bounded by

$$0 \leq R_N \leq \int_N^{\infty} f(x)\,dx.$$

54. Using a Remainder Show that the result of Exercise 53 can be written as

$$\sum_{n=1}^{N} a_n \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{N} a_n + \int_N^{\infty} f(x)\,dx.$$

Approximating a Sum **In Exercises 55–60, use the result of Exercise 53 to approximate the sum of the convergent series using the indicated number of terms. Include an estimate of the maximum error for your approximation.**

55. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, five terms

56. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$, six terms

57. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, ten terms

58. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^3}$, ten terms

59. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$, four terms

60. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$, four terms

Finding a Value **In Exercises 61–64, use the result of Exercise 53 to find N such that $R_N \leq 0.001$ for the convergent series.**

61. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

62. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

63. $\displaystyle\sum_{n=1}^{\infty} e^{-n/2}$

64. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

65. Comparing Series

(a) Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$ converges and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

(b) Compare the first five terms of each series in part (a).

(c) Find $n > 3$ such that $\displaystyle\frac{1}{n^{1.1}} < \frac{1}{n \ln n}$.

66. Using a p-Series Ten terms are used to approximate a convergent p-series. Therefore, the remainder is a function of p and is

$$0 \leq R_{10}(p) \leq \int_{10}^{\infty} \frac{1}{x^p}\,dx, \quad p > 1.$$

(a) Perform the integration in the inequality.

(b) Use a graphing utility to represent the inequality graphically.

(c) Identify any asymptotes of the error function and interpret their meaning.

67. Euler's Constant Let

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(a) Show that $\ln(n + 1) \le S_n \le 1 + \ln n$.

(b) Show that the sequence $\{a_n\} = \{S_n - \ln n\}$ is bounded.

(c) Show that the sequence $\{a_n\}$ is decreasing.

(d) Show that a_n converges to a limit γ (called Euler's constant).

(e) Approximate γ using a_{100}.

68. Finding a Sum Find the sum of the series

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right).$$

69. Using a Series Consider the series $\displaystyle\sum_{n=2}^{\infty} x^{\ln n}$.

(a) Determine the convergence or divergence of the series for $x = 1$.

(b) Determine the convergence or divergence of the series for $x = 1/e$.

(c) Find the positive values of x for which the series converges.

70. Riemann Zeta Function The **Riemann zeta function** for real numbers is defined for all x for which the series

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$$

converges. Find the domain of the function.

Review **In Exercises 71–82, determine the convergence or divergence of the series.**

71. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n - 2}$

72. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$

73. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$

74. $\displaystyle 3\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

75. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

76. $\displaystyle\sum_{n=0}^{\infty} (1.042)^n$

77. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}}$

78. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right)$

79. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

80. $\displaystyle\sum_{n=2}^{\infty} \ln n$

81. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

82. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

SECTION PROJECT

The Harmonic Series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is one of the most important series in this chapter. Even though its terms tend to zero as n increases,

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

the harmonic series diverges. In other words, even though the terms are getting smaller and smaller, the sum "adds up to infinity."

(a) One way to show that the harmonic series diverges is attributed to James Bernoulli. He grouped the terms of the harmonic series as follows:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{>\frac{1}{2}} + \underbrace{\frac{1}{17} + \cdots + \frac{1}{32}}_{>\frac{1}{2}} + \cdots$$

Write a short paragraph explaining how you can use this grouping to show that the harmonic series diverges.

(b) Use the proof of the Integral Test, Theorem 9.10, to show that

$$\ln(n + 1) \le 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le 1 + \ln n.$$

(c) Use part (b) to determine how many terms M you would need so that

$$\sum_{n=1}^{M} \frac{1}{n} > 50.$$

(d) Show that the sum of the first million terms of the harmonic series is less than 15.

(e) Show that the following inequalities are valid.

$$\ln\frac{21}{10} \le \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{20} \le \ln\frac{20}{9}$$

$$\ln\frac{201}{100} \le \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{200} \le \ln\frac{200}{99}$$

(f) Use the inequalities in part (e) to find the limit

$$\lim_{m\to\infty} \sum_{n=m}^{2m} \frac{1}{n}.$$

9.4 Comparisons of Series

- **Use the Direct Comparison Test to determine whether a series converges or diverges.**
- **Use the Limit Comparison Test to determine whether a series converges or diverges.**

Direct Comparison Test

For the convergence tests developed so far, the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied. A slight deviation from these special characteristics can make a test nonapplicable. For example, in the pairs listed below, the second series cannot be tested by the same convergence test as the first series, even though it is similar to the first.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is not.

3. $a_n = \dfrac{n}{(n^2+3)^2}$ is easily integrated, but $b_n = \dfrac{n^2}{(n^2+3)^2}$ is not.

In this section, you will study two additional tests for positive-term series. These two tests greatly expand the variety of series you are able to test for convergence or divergence. They allow you to *compare* a series having complicated terms with a simpler series whose convergence or divergence is known.

REMARK As stated, the Direct Comparison Test requires that $0 < a_n \leq b_n$ for all n. Because the convergence of a series is not dependent on its first several terms, you could modify the test to require only that $0 < a_n \leq b_n$ for all n greater than some integer N.

THEOREM 9.12 Direct Comparison Test

Let $0 < a_n \leq b_n$ for all n.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Proof To prove the first property, let $L = \displaystyle\sum_{n=1}^{\infty} b_n$ and let

$$S_n = a_1 + a_2 + \cdot \cdot \cdot + a_n.$$

Because $0 < a_n \leq b_n$, the sequence $S_1, S_2, S_3, \ldots$ is nondecreasing and bounded above by L; so, it must converge. Because

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$$

it follows that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. The second property is logically equivalent to the first.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

FOR FURTHER INFORMATION Is the Direct Comparison Test just for nonnegative series? To read about the generalization of this test to real series, see the article "The Comparison Test—Not Just for Nonnegative Series" by Michele Longo and Vincenzo Valori in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

EXAMPLE 1 Using the Direct Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}.$$

Solution This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{3^n}. \qquad \text{Convergent geometric series}$$

Term-by-term comparison yields

$$a_n = \frac{1}{2 + 3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1.$$

So, by the Direct Comparison Test, the series converges.

EXAMPLE 2 Using the Direct Comparison Test

See LarsonCalculus.com for an interactive version of this type of example.

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}.$$

Solution This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}. \qquad \text{Divergent } p\text{-series}$$

Term-by-term comparison yields

$$\frac{1}{2 + \sqrt{n}} \leq \frac{1}{\sqrt{n}}, \quad n \geq 1$$

which *does not* meet the requirements for divergence. (Remember that when term-by-term comparison reveals a series that is *less* than a divergent series, the Direct Comparison Test tells you nothing.) Still expecting the series to diverge, you can compare the series with

$$\sum_{n=1}^{\infty} \frac{1}{n}. \qquad \text{Divergent harmonic series}$$

In this case, term-by-term comparison yields

$$a_n = \frac{1}{n} \leq \frac{1}{2 + \sqrt{n}} = b_n, \quad n \geq 4$$

and, by the Direct Comparison Test, the given series diverges. To verify the last inequality, try showing that

$$2 + \sqrt{n} \leq n$$

whenever $n \geq 4$.

Remember that both parts of the Direct Comparison Test require that $0 < a_n \leq b_n$. Informally, the test says the following about the two series with nonnegative terms.

1. If the "larger" series converges, then the "smaller" series must also converge.
2. If the "smaller" series diverges, then the "larger" series must also diverge.

Limit Comparison Test

Sometimes a series closely resembles a p-series or a geometric series, yet you cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test. Under these circumstances, you may be able to apply a second comparison test, called the **Limit Comparison Test.**

REMARK As with the Direct Comparison Test, the Limit Comparison Test could be modified to require only that a_n and b_n be positive for all n greater than some integer N.

THEOREM 9.13 Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is *finite and positive*, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

Proof Because $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

there exists $N > 0$ such that

$$0 < \frac{a_n}{b_n} < L + 1, \quad \text{for } n \geq N.$$

This implies that

$$0 < a_n < (L + 1)b_n.$$

So, by the Direct Comparison Test, the convergence of $\Sigma\, b_n$ implies the convergence of $\Sigma\, a_n$. Similarly, the fact that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L}$$

can be used to show that the convergence of $\Sigma\, a_n$ implies the convergence of $\Sigma\, b_n$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 3 Using the Limit Comparison Test

Show that the general harmonic series below diverges.

$$\sum_{n=1}^{\infty} \frac{1}{an + b}, \quad a > 0, \quad b > 0$$

Solution By comparison with

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{Divergent harmonic series}$$

you have

$$\lim_{n\to\infty} \frac{1/(an + b)}{1/n} = \lim_{n\to\infty} \frac{n}{an + b} = \frac{1}{a}.$$

Because this limit is greater than 0, you can conclude from the Limit Comparison Test that the series diverges.

The Limit Comparison Test works well for comparing a "messy" algebraic series with a p-series. In choosing an appropriate p-series, you must choose one with an nth term of the same magnitude as the nth term of the given series.

Given Series	Comparison Series	Conclusion
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$	Both series converge.
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	Both series diverge.
$\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 10}{4n^5 + n^3}$	$\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$	Both series converge.

In other words, when choosing a series for comparison, you can disregard all but the *highest powers of n* in both the numerator and the denominator.

EXAMPLE 4 Using the Limit Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}.$$

Solution Disregarding all but the highest powers of n in the numerator and the denominator, you can compare the series with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}. \qquad \text{Convergent } p\text{-series}$$

Because

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2 + 1}\right)\left(\frac{n^{3/2}}{1}\right)\\ &= \lim_{n\to\infty} \frac{n^2}{n^2 + 1}\\ &= 1\end{aligned}$$

you can conclude by the Limit Comparison Test that the series converges.

EXAMPLE 5 Using the Limit Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n2^n}{4n^3 + 1}.$$

Solution A reasonable comparison would be with the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}. \qquad \text{Divergent series}$$

Note that this series diverges by the nth-Term Test. From the limit

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(\frac{n2^n}{4n^3 + 1}\right)\left(\frac{n^2}{2^n}\right)\\ &= \lim_{n\to\infty} \frac{1}{4 + (1/n^3)}\\ &= \frac{1}{4}\end{aligned}$$

you can conclude that the series diverges.

9.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

1. Graphical Analysis The figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

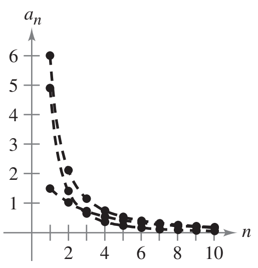
$$\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}, \quad \sum_{n=1}^{\infty} \frac{6}{n^{3/2}+3}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2+0.5}}$$

(a) Identify the series in each figure.

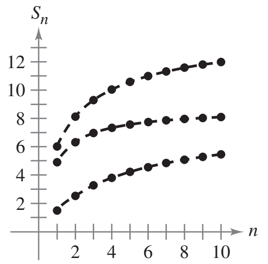
(b) Which series is a p-series? Does it converge or diverge?

(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?

(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.



Graphs of terms



Graphs of partial sums

2. Graphical Analysis The figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

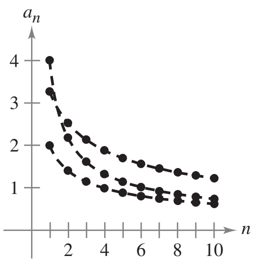
$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}-0.5}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{4}{\sqrt{n+0.5}}$$

(a) Identify the series in each figure.

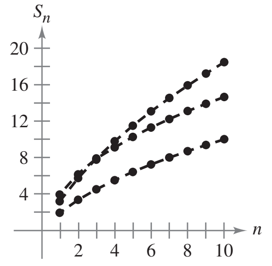
(b) Which series is a p-series? Does it converge or diverge?

(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?

(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.



Graphs of terms



Graphs of partial sums

Using the Direct Comparison Test **In Exercises 3–12, use the Direct Comparison Test to determine the convergence or divergence of the series.**

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ **4.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ **6.** $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n+3}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$ **8.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

9. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ **10.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

11. $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$ **12.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$

Using the Limit Comparison Test **In Exercises 13–22, use the Limit Comparison Test to determine the convergence or divergence of the series.**

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ **14.** $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ **16.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$ **18.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ **20.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}, \quad k > 2$ **22.** $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$

Determining Convergence or Divergence **In Exercises 23–30, test for convergence or divergence, using each test at least once. Identify which test was used.**

(a) nth-Term Test **(b) Geometric Series Test**

(c) p-Series Test **(d) Telescoping Series Test**

(e) Integral Test **(f) Direct Comparison Test**

(g) Limit Comparison Test

23. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ **24.** $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ **26.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-8}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$ **28.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ **30.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

31. Using the Limit Comparison Test Use the Limit Comparison Test with the harmonic series to show that the series $\sum a_n$ (where $0 < a_n < a_{n-1}$) diverges when $\displaystyle\lim_{n\to\infty} na_n$ is finite and nonzero.

32. Proof Prove that, if $P(n)$ and $Q(n)$ are polynomials of degree j and k, respectively, then the series

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

converges if $j < k - 1$ and diverges if $j \geq k - 1$.

Determining Convergence or Divergence **In Exercises 33–36, use the polynomial test given in Exercise 32 to determine whether the series converges or diverges.**

33. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots$

34. $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

Verifying Divergence **In Exercises 37 and 38, use the divergence test given in Exercise 31 to show that the series diverges.**

37. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 1}{4n^3 + 2}$

Determining Convergence or Divergence **In Exercises 39–42, determine the convergence or divergence of the series.**

39. $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \frac{1}{800} + \cdots$

40. $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \frac{1}{230} + \cdots$

41. $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} + \cdots$

42. $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots$

WRITING ABOUT CONCEPTS

43. Using Series Review the results of Exercises 39–42. Explain why careful analysis is required to determine the convergence or divergence of a series and why only considering the magnitudes of the terms of a series could be misleading.

44. Direct Comparison Test State the Direct Comparison Test and give an example of its use.

45. Limit Comparison Test State the Limit Comparison Test and give an example of its use.

46. Comparing Series It appears that the terms of the series

$$\frac{1}{1000} + \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \cdots$$

are less than the corresponding terms of the convergent series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

If the statement above is correct, then the first series converges. Is this correct? Why or why not? Make a statement about how the divergence or convergence of a series is affected by the inclusion or exclusion of the first finite number of terms.

47. Using a Series Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

(a) Verify that the series converges.

(b) Use a graphing utility to complete the table.

n	5	10	20	50	100
S_n					

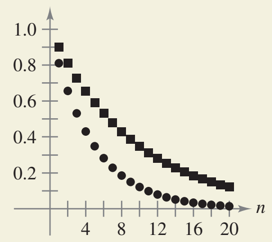
(c) The sum of the series is $\pi^2/8$. Find the sum of the series

$$\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2}.$$

(d) Use a graphing utility to find the sum of the series

$$\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2}.$$

48. HOW DO YOU SEE IT? The figure shows the first 20 terms of the convergent series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the first 20 terms of the series $\displaystyle\sum_{n=1}^{\infty} a_n^2$. Identify the two series and explain your reasoning in making the selection.



True or False? **In Exercises 49–54, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

49. If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

50. If $0 < a_{n+10} \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

51. If $a_n + b_n \leq c_n$ and $\displaystyle\sum_{n=1}^{\infty} c_n$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ both converge. (Assume that the terms of all three series are positive.)

52. If $a_n \leq b_n + c_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty} b_n$ and $\displaystyle\sum_{n=1}^{\infty} c_n$ both diverge. (Assume that the terms of all three series are positive.)

53. If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

54. If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

55. Proof Prove that if the nonnegative series

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

converge, then so does the series $\sum_{n=1}^{\infty} a_n b_n$.

56. Proof Use the result of Exercise 55 to prove that if the nonnegative series $\sum_{n=1}^{\infty} a_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n^2$.

57. Finding Series Find two series that demonstrate the result of Exercise 55.

58. Finding Series Find two series that demonstrate the result of Exercise 56.

59. Proof Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. Prove that if $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\Sigma\, b_n$ converges, $\Sigma\, a_n$ also converges.

60. Proof Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. Prove that if $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\Sigma\, b_n$ diverges, $\Sigma\, a_n$ also diverges.

61. Verifying Convergence Use the result of Exercise 59 to show that each series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,\pi^n}$

62. Verifying Divergence Use the result of Exercise 60 to show that each series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

63. Proof Suppose that $\Sigma\, a_n$ is a series with positive terms. Prove that if $\Sigma\, a_n$ converges, then $\Sigma \sin a_n$ also converges.

64. Proof Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$$

converges.

65. Comparing Series Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n}}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$.

PUTNAM EXAM CHALLENGE

66. Is the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{(n+1)/n}}$ convergent? Prove your statement.

67. Prove that if $\displaystyle\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\displaystyle\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

SECTION PROJECT

Solera Method

Most wines are produced entirely from grapes grown in a single year. Sherry, however, is a complex mixture of older wines with new wines. This is done with a sequence of barrels (called a solera) stacked on top of each other, as shown in the photo.

The oldest wine is in the bottom tier of barrels, and the newest is in the top tier. Each year, half of each barrel in the bottom tier is bottled as sherry. The bottom barrels are then refilled with the wine from the barrels above. This process is repeated throughout the solera, with new wine being added to the top barrels.

A mathematical model for the amount of n-year-old wine that is removed from a solera (with k tiers) each year is

$$f(n, k) = \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n+1}, \quad k \le n.$$

(a) Consider a solera that has five tiers, numbered $k = 1, 2, 3, 4,$ and 5. In 1995 ($n = 0$), half of each barrel in the top tier (tier 1) was refilled with new wine. How much of this wine was removed from the solera in 1996? In 1997? In 1998? . . . In 2010? During which year(s) was the greatest amount of the 1995 wine removed from the solera?

(b) In part (a), let a_n be the amount of 1995 wine that is removed from the solera in year n. Evaluate

$$\sum_{n=0}^{\infty} a_n.$$

FOR FURTHER INFORMATION See the article "Finding Vintage Concentrations in a Sherry Solera" by Rhodes Peele and John T. MacQueen in the *UMAP Modules.*

Squareplum/Shutterstock.com

9.5 Alternating Series

- **Use the Alternating Series Test to determine whether an infinite series converges.**
- **Use the Alternating Series Remainder to approximate the sum of an alternating series.**
- **Classify a convergent series as absolutely or conditionally convergent.**
- **Rearrange an infinite series to obtain a different sum.**

Alternating Series

So far, most series you have dealt with have had positive terms. In this section and the next section, you will study series that contain both positive and negative terms. The simplest such series is an **alternating series,** whose terms alternate in sign. For example, the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$
$$= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$$

is an *alternating geometric series* with $r = -\frac{1}{2}$. Alternating series occur in two ways: either the odd terms are negative or the even terms are negative.

THEOREM 9.14 Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge when the two conditions listed below are met.

1. $\lim\limits_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$, for all n

REMARK The second condition in the Alternating Series Test can be modified to require only that $0 < a_{n+1} \le a_n$ for all n greater than some integer N.

Proof Consider the alternating series $\Sigma(-1)^{n+1}a_n$. For this series, the partial sum (where $2n$ is even)

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{2n-1} - a_{2n})$$

has all nonnegative terms, and therefore $\{S_{2n}\}$ is a nondecreasing sequence. But you can also write

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

which implies that $S_{2n} \le a_1$ for every integer n. So, $\{S_{2n}\}$ is a bounded, nondecreasing sequence that converges to some value L. Because $S_{2n-1} - a_{2n} = S_{2n}$ and $a_{2n} \to 0$, you have

$$\lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n}$$
$$= L + \lim_{n\to\infty} a_{2n}$$
$$= L.$$

Because both S_{2n} and S_{2n-1} converge to the same limit L, it follows that $\{S_n\}$ also converges to L. Consequently, the given alternating series converges.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1 **Using the Alternating Series Test**

> **REMARK** The series in Example 1 is called the *alternating harmonic series.* More is said about this series in Example 8.

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}.$$

Solution Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. So, the first condition of Theorem 9.14 is satisfied. Also note that the second condition of Theorem 9.14 is satisfied because

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

for all n. So, applying the Alternating Series Test, you can conclude that the series converges.

EXAMPLE 2 **Using the Alternating Series Test**

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}.$$

Solution To apply the Alternating Series Test, note that, for $n \geq 1$,

$$\begin{aligned}\frac{1}{2} &\leq \frac{n}{n+1}\\ \frac{2^{n-1}}{2^n} &\leq \frac{n}{n+1}\\ (n+1)2^{n-1} &\leq n2^n\\ \frac{n+1}{2^n} &\leq \frac{n}{2^{n-1}}.\end{aligned}$$

So, $a_{n+1} = (n+1)/2^n \leq n/2^{n-1} = a_n$ for all n. Furthermore, by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x}{2^{x-1}} = \lim_{x\to\infty} \frac{1}{2^{x-1}(\ln 2)} = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{n}{2^{n-1}} = 0.$$

Therefore, by the Alternating Series Test, the series converges.

EXAMPLE 3 **When the Alternating Series Test Does Not Apply**

> **REMARK** In Example 3(a), remember that whenever a series does not pass the first condition of the Alternating Series Test, you can use the nth-Term Test for Divergence to conclude that the series diverges.

a. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots$$

passes the second condition of the Alternating Series Test because $a_{n+1} \leq a_n$ for all n. You cannot apply the Alternating Series Test, however, because the series does not pass the first condition. In fact, the series diverges.

b. The alternating series

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \cdots$$

passes the first condition because a_n approaches 0 as $n \to \infty$. You cannot apply the Alternating Series Test, however, because the series does not pass the second condition. To conclude that the series diverges, you can argue that S_{2N} equals the Nth partial sum of the divergent harmonic series. This implies that the sequence of partial sums diverges. So, the series diverges.

Alternating Series Remainder

For a convergent alternating series, the partial sum S_N can be a useful approximation for the sum S of the series. The error involved in using $S \approx S_N$ is the remainder $R_N = S - S_N$.

THEOREM 9.15 Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \le a_{N+1}.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 4 Approximating the Sum of an Alternating Series

See LarsonCalculus.com for an interactive version of this type of example.

Approximate the sum of the series by its first six terms.

$$\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n!}\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdot \cdot \cdot$$

Solution The series converges by the Alternating Series Test because

$$\frac{1}{(n+1)!} \le \frac{1}{n!} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n!} = 0.$$

The sum of the first six terms is

$$S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144} \approx 0.63194$$

and, by the Alternating Series Remainder, you have

$$|S - S_6| = |R_6| \le a_7 = \frac{1}{5040} \approx 0.0002.$$

So, the sum S lies between $0.63194 - 0.0002$ and $0.63194 + 0.0002$, and you have $0.63174 \le S \le 0.63214$.

TECHNOLOGY Later, using the techniques in Section 9.10, you will be able to show that the series in Example 4 converges to

$$\frac{e-1}{e} \approx 0.63212.$$

(See Section 9.10, Exercise 58.) For now, try using a graphing utility to obtain an approximation of the sum of the series. How many terms do you need to obtain an approximation that is within 0.00001 unit of the actual sum?

EXAMPLE 5 Finding the Number of Terms

Determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

Solution By Theorem 9.15, you know that

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)^4}.$$

For an error of less than 0.001, N must satisfy the inequality $1/(N+1)^4 < 0.001$.

$$\frac{1}{(N+1)^4} < 0.001 \quad \Rightarrow \quad (N+1)^4 > 1000 \quad \Rightarrow \quad N > \sqrt[4]{1000} - 1 \approx 4.6$$

So, you will need at least 5 terms. Using 5 terms, the sum is $S \approx S_5 \approx 0.94754$, which has an error of less than 0.001.

Absolute and Conditional Convergence

Occasionally, a series may have both positive and negative terms and not be an alternating series. For instance, the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

has both positive and negative terms, yet it is not an alternating series. One way to obtain some information about the convergence of this series is to investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|.$$

By direct comparison, you have $|\sin n| \leq 1$ for all n, so

$$\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}, \quad n \geq 1.$$

Therefore, by the Direct Comparison Test, the series $\sum \left|\frac{\sin n}{n^2}\right|$ converges. The next theorem tells you that the original series also converges.

THEOREM 9.16 Absolute Convergence

If the series $\Sigma\, |a_n|$ converges, then the series $\Sigma\, a_n$ also converges.

Proof Because $0 \leq a_n + |a_n| \leq 2|a_n|$ for all n, the series

$$\sum_{n=1}^{\infty} (a_n + |a_n|)$$

converges by comparison with the convergent series

$$\sum_{n=1}^{\infty} 2|a_n|.$$

Furthermore, because $a_n = (a_n + |a_n|) - |a_n|$, you can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

where both series on the right converge. So, it follows that $\Sigma\, a_n$ converges.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The converse of Theorem 9.16 is not true. For instance, the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called **conditional.**

Definitions of Absolute and Conditional Convergence

1. The series $\Sigma\, a_n$ is **absolutely convergent** when $\Sigma\, |a_n|$ converges.
2. The series $\Sigma\, a_n$ is **conditionally convergent** when $\Sigma\, a_n$ converges but $\Sigma\, |a_n|$ diverges.

EXAMPLE 6 Absolute and Conditional Convergence

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{2^n} = \frac{0!}{2^0} - \frac{1!}{2^1} + \frac{2!}{2^2} - \frac{3!}{2^3} + \cdots$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdots$

Solution

a. This is an alternating series, but the Alternating Series Test does not apply because the limit of the nth term is not zero. By the nth-Term Test for Divergence, however, you can conclude that this series diverges.

b. This series can be shown to be convergent by the Alternating Series Test. Moreover, because the p-series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

diverges, the given series is *conditionally* convergent.

EXAMPLE 7 Absolute and Conditional Convergence

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} = -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} - \cdots$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = -\frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \cdots$

Solution

a. This is *not* an alternating series (the signs change in pairs). However, note that

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n(n+1)/2}}{3^n}\right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

is a convergent geometric series, with

$$r = \frac{1}{3}.$$

Consequently, by Theorem 9.16, you can conclude that the given series is *absolutely* convergent (and therefore convergent).

b. In this case, the Alternating Series Test indicates that the series converges. However, the series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\ln(n+1)}\right| = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots$$

diverges by direct comparison with the terms of the harmonic series. Therefore, the given series is *conditionally* convergent.

FOR FURTHER INFORMATION To read more about the convergence of alternating harmonic series, see the article "Almost Alternating Harmonic Series" by Curtis Feist and Ramin Naimi in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

Rearrangement of Series

A finite sum such as

$$1 + 3 - 2 + 5 - 4$$

can be rearranged without changing the value of the sum. This is not necessarily true of an infinite series—it depends on whether the series is absolutely convergent or conditionally convergent.

1. If a series is *absolutely convergent,* then its terms can be rearranged in any order without changing the sum of the series.
2. If a series is *conditionally convergent,* then its terms can be rearranged to give a different sum.

The second case is illustrated in Example 8.

EXAMPLE 8 Rearrangement of a Series

The alternating harmonic series converges to ln 2. That is,

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

(See Exercise 55, Section 9.10.)

Rearrange the series to produce a different sum.

Solution Consider the rearrangement below.

$$\begin{aligned}
1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \cdots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots\right) \\
&= \frac{1}{2}(\ln 2)
\end{aligned}$$

By rearranging the terms, you obtain a sum that is half the original sum.

FOR FURTHER INFORMATION Georg Friedrich Bernhard Riemann (1826–1866) proved that if $\sum a_n$ is conditionally convergent and S is any real number, then the terms of the series can be rearranged to converge to S. For more on this topic, see the article "Riemann's Rearrangement Theorem" by Stewart Galanor in *Mathematics Teacher.* To view this article, go to *MathArticles.com.*

Exploration

In Example 8, you learned that the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges to $\ln 2 \approx 0.693$. Rearrangement of the terms of the series produces a different sum, $\frac{1}{2}\ln 2 \approx 0.347$.

In this exploration, you will rearrange the terms of the alternating harmonic series in such a way that two positive terms follow each negative term. That is,

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \cdots.$$

Now calculate the partial sums S_4, S_7, S_{10}, S_{13}, S_{16}, and S_{19}. Then estimate the sum of this series to three decimal places.

9.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Numerical and Graphical Analysis In Exercises 1–4, explore the Alternating Series Remainder.

(a) Use a graphing utility to find the indicated partial sum S_n and complete the table.

n	1	2	3	4	5	6	7	8	9	10
S_n										

(b) Use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum.

(c) What pattern exists between the plot of the successive points in part (b) relative to the horizontal line representing the sum of the series? Do the distances between the successive points and the horizontal line increase or decrease?

(d) Discuss the relationship between the answers in part (c) and the Alternating Series Remainder as given in Theorem 9.15.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin 1$

Determining Convergence or Divergence In Exercises 5–26, determine the convergence or divergence of the series.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3n+2}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{3^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{e^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}(5n-1)}{4n+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n} n}{\ln(n+1)}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

17. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1\cdot 3\cdot 5\cdots(2n-1)}$

24. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 4\cdot 7\cdots(3n-2)}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n - e^{-n}} = \sum_{n=1}^{\infty} (-1)^{n+1}\operatorname{csch} n$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} (-1)^{n+1}\operatorname{sech} n$

Approximating the Sum of an Alternating Series In Exercises 27–30, approximate the sum of the series by using the first six terms. (See Example 4.)

27. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}5}{n!}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{\ln(n+1)}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n^3}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{3^n}$

Finding the Number of Terms In Exercises 31–36, use Theorem 9.15 to determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$

35. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!}$

Determining Absolute and Conditional Convergence In Exercises 37–54, determine whether the series converges absolutely or conditionally, or diverges.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2^n}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{(n+1)^2}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

45. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n}}{n\ln n}$

46. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n}e^{-n^2}$

47. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n}n}{n^3-5}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

49. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

50. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{\sqrt{n+4}}$

51. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

52. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\arctan n$

53. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n}$

WRITING ABOUT CONCEPTS

55. Alternating Series Define an alternating series.

56. Alternating Series Test State the Alternating Series Test.

57. Alternating Series Remainder Give the remainder after N terms of a convergent alternating series.

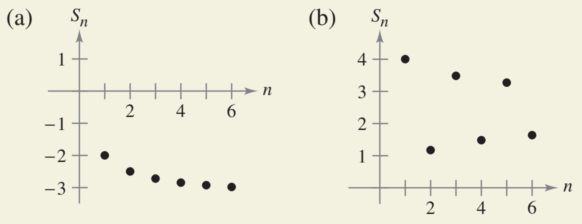
58. Absolute and Conditional Convergence In your own words, state the difference between absolute and conditional convergence of an alternating series.

59. Think About It Do you agree with the following statements? Why or why not?

(a) If both $\sum a_n$ and $\sum (-a_n)$ converge, then $\sum |a_n|$ converges.

(b) If $\sum a_n$ diverges, then $\sum |a_n|$ diverges.

60. HOW DO YOU SEE IT? The graphs of the sequences of partial sums of two series are shown in the figures. Which graph represents the partial sums of an alternating series? Explain.



True or False? **In Exercises 61 and 62, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

61. For the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

the partial sum S_{100} is an overestimate of the sum of the series.

62. If $\sum a_n$ and $\sum b_n$ both converge, then $\sum a_n b_n$ converges.

Finding Values **In Exercises 63 and 64, find the values of p for which the series converges.**

63. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(\frac{1}{n^p}\right)$

64. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(\frac{1}{n+p}\right)$

65. Proof Prove that if $\sum |a_n|$ converges, then $\sum a_n^2$ converges. Is the converse true? If not, give an example that shows it is false.

66. Finding a Series Use the result of Exercise 63 to give an example of an alternating p-series that converges, but whose corresponding p-series diverges.

67. Finding a Series Give an example of a series that demonstrates the statement you proved in Exercise 65.

68. Finding Values Find all values of x for which the series $\sum (x^n/n)$ (a) converges absolutely and (b) converges conditionally.

Using a Series **In Exercises 69 and 70, use the given series.**

(a) Does the series meet the conditions of Theorem 9.14? Explain why or why not.

(b) Does the series converge? If so, what is the sum?

69. $\displaystyle\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \cdots + \frac{1}{2^n} - \frac{1}{3^n} + \cdots$

70. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n,\ a_n = \begin{cases} \dfrac{1}{\sqrt{n}}, & \text{if } n \text{ is odd} \\ \dfrac{1}{n^3}, & \text{if } n \text{ is even} \end{cases}$

Review **In Exercises 71–80, test for convergence or divergence and identify the test used.**

71. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n^{3/2}}$

72. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+5}$

73. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

74. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

75. $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{7}{8}\right)^n$

76. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$

77. $\displaystyle\sum_{n=1}^{\infty} 100e^{-n/2}$

78. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4}$

79. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{3n^2-1}$

80. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

81. Describing an Error The following argument, that $0 = 1$, is *incorrect*. Describe the error.

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + \cdots \\ &= 1 \end{aligned}$$

PUTNAM EXAM CHALLENGE

82. Assume as known the (true) fact that the alternating harmonic series

(1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$

is convergent, and denote its sum by s. Rearrange the series (1) as follows:

(2) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots.$

Assume as known the (true) fact that the series (2) is also convergent, and denote its sum by S. Denote by s_k, S_k the kth partial sum of the series (1) and (2), respectively. Prove the following statements.

(i) $S_{3n} = s_{4n} + \frac{1}{2}s_{2n}$, (ii) $S \neq s$

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

9.6 The Ratio and Root Tests

- **Use the Ratio Test to determine whether a series converges or diverges.**
- **Use the Root Test to determine whether a series converges or diverges.**
- **Review the tests for convergence and divergence of an infinite series.**

The Ratio Test

This section begins with a test for absolute convergence—the **Ratio Test.**

THEOREM 9.17 Ratio Test

Let $\Sigma\, a_n$ be a series with nonzero terms.

1. The series $\Sigma\, a_n$ converges absolutely when $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$
2. The series $\Sigma\, a_n$ diverges when $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty.$
3. The Ratio Test is inconclusive when $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$

Proof To prove Property 1, assume that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r < 1$$

and choose R such that $0 \le r < R < 1$. By the definition of the limit of a sequence, there exists some $N > 0$ such that $|a_{n+1}/a_n| < R$ for all $n > N$. Therefore, you can write the following inequalities.

$$\begin{aligned}
|a_{N+1}| &< |a_N|R \\
|a_{N+2}| &< |a_{N+1}|R < |a_N|R^2 \\
|a_{N+3}| &< |a_{N+2}|R < |a_{N+1}|R^2 < |a_N|R^3 \\
&\vdots
\end{aligned}$$

The geometric series $\displaystyle\sum_{n=1}^{\infty} |a_N|R^n = |a_N|R + |a_N|R^2 + \cdot\cdot\cdot + |a_N|R^n + \cdot\cdot\cdot$ converges, and so, by the Direct Comparison Test, the series

$$\sum_{n=1}^{\infty} |a_{N+n}| = |a_{N+1}| + |a_{N+2}| + \cdot\cdot\cdot + |a_{N+n}| + \cdot\cdot\cdot$$

also converges. This in turn implies that the series $\Sigma\, |a_n|$ converges, because discarding a finite number of terms $(n = N - 1)$ does not affect convergence. Consequently, by Theorem 9.16, the series $\Sigma\, a_n$ converges absolutely. The proof of Property 2 is similar and is left as an exercise (see Exercise 99).

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The fact that the Ratio Test is inconclusive when $|a_{n+1}/a_n| \to 1$ can be seen by comparing the two series $\Sigma\,(1/n)$ and $\Sigma\,(1/n^2)$. The first series diverges and the second one converges, but in both cases

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$$

Although the Ratio Test is not a cure for all ills related to testing for convergence, it is particularly useful for series that *converge rapidly.* Series involving factorials or exponentials are frequently of this type.

EXAMPLE 1 **Using the Ratio Test**

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Solution Because

$$a_n = \frac{2^n}{n!}$$

you can write the following.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!}\right] \\
&= \lim_{n\to\infty} \left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right] \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0 < 1
\end{aligned}$$

This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1.

REMARK A step frequently used in applications of the Ratio Test involves simplifying quotients of factorials. In Example 1, for instance, notice that

$$\frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1}.$$

EXAMPLE 2 **Using the Ratio Test**

Determine whether each series converges or diverges.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

a. This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[(n+1)^2\left(\frac{2^{n+2}}{3^{n+1}}\right)\left(\frac{3^n}{n^2 2^{n+1}}\right)\right] \\
&= \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2} \\
&= \frac{2}{3} < 1
\end{aligned}$$

b. This series diverges because the limit of $|a_{n+1}/a_n|$ is greater than 1.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{(n+1)!}\left(\frac{n!}{n^n}\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{(n+1)}\left(\frac{1}{n^n}\right)\right] \\
&= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \\
&= e > 1
\end{aligned}$$

EXAMPLE 3 A Failure of the Ratio Test

See LarsonCalculus.com for an interactive version of this type of example.

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}.$$

Solution The limit of $|a_{n+1}/a_n|$ is equal to 1.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\left(\frac{\sqrt{n+1}}{n+2}\right)\left(\frac{n+1}{\sqrt{n}}\right)\right] \\ &= \lim_{n\to\infty}\left[\sqrt{\frac{n+1}{n}}\left(\frac{n+1}{n+2}\right)\right] \\ &= \sqrt{1}(1) \\ &= 1\end{aligned}$$

REMARK The Ratio Test is also inconclusive for any p-series.

So, the Ratio Test is inconclusive. To determine whether the series converges, you need to try a different test. In this case, you can apply the Alternating Series Test. To show that $a_{n+1} \le a_n$, let

$$f(x) = \frac{\sqrt{x}}{x+1}.$$

Then the derivative is

$$f'(x) = \frac{-x+1}{2\sqrt{x}(x+1)^2}.$$

Because the derivative is negative for $x > 1$, you know that f is a decreasing function. Also, by L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{x}}{x+1} &= \lim_{x\to\infty}\frac{1/\left(2\sqrt{x}\right)}{1} \\ &= \lim_{x\to\infty}\frac{1}{2\sqrt{x}} \\ &= 0.\end{aligned}$$

Therefore, by the Alternating Series Test, the series converges.

The series in Example 3 is *conditionally convergent*. This follows from the fact that the series

$$\sum_{n=1}^{\infty} |a_n|$$

diverges $\left(\text{by the Limit Comparison Test with } \Sigma\, 1/\sqrt{n}\right)$, but the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

TECHNOLOGY A graphing utility can reinforce the conclusion that the series in Example 3 converges *conditionally*. By adding the first 100 terms of the series, you obtain a sum of about -0.2. (The sum of the first 100 terms of the series $\Sigma\,|a_n|$ is about 17.)

The Root Test

The next test for convergence or divergence of series works especially well for series involving nth powers. The proof of this theorem is similar to the proof given for the Ratio Test, and is left as an exercise (see Exercise 100).

REMARK The Root Test is always inconclusive for any p-series.

THEOREM 9.18 Root Test

1. The series $\Sigma\, a_n$ converges absolutely when $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.
2. The series $\Sigma\, a_n$ diverges when $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

EXAMPLE 4 Using the Root Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}.$$

Solution You can apply the Root Test as follows.

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} \\
&= \lim_{n\to\infty} \frac{e^{2n/n}}{n^{n/n}} \\
&= \lim_{n\to\infty} \frac{e^2}{n} \\
&= 0 < 1
\end{aligned}$$

Because this limit is less than 1, you can conclude that the series converges absolutely (and therefore converges).

To see the usefulness of the Root Test for the series in Example 4, try applying the Ratio Test to that series. When you do this, you obtain the following.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{e^{2(n+1)}}{(n+1)^{n+1}} \div \frac{e^{2n}}{n^n}\right] \\
&= \lim_{n\to\infty} \left[\frac{e^{2(n+1)}}{(n+1)^{n+1}} \cdot \frac{n^n}{e^{2n}}\right] \\
&= \lim_{n\to\infty} e^2 \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} e^2 \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) \\
&= 0
\end{aligned}$$

Note that this limit is not as easily evaluated as the limit obtained by the Root Test in Example 4.

FOR FURTHER INFORMATION For more information on the usefulness of the Root Test, see the article "*N*! and the Root Test" by Charles C. Mumma II in *The American Mathematical Monthly.* To view this article, go to *MathArticles.com.*

Strategies for Testing Series

You have now studied 10 tests for determining the convergence or divergence of an infinite series. (See the summary in the table on the next page.) Skill in choosing and applying the various tests will come only with practice. Below is a set of guidelines for choosing an appropriate test.

GUIDELINES FOR TESTING A SERIES FOR CONVERGENCE OR DIVERGENCE

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

In some instances, more than one test is applicable. However, your objective should be to learn to choose the most efficient test.

EXAMPLE 5 Applying the Strategies for Testing Series

Determine the convergence or divergence of each series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ **c.** $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$ **e.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ **f.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

g. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Solution

a. For this series, the limit of the nth term is not 0 $\left(a_n \to \frac{1}{3} \text{ as } n \to \infty\right)$. So, by the nth-Term Test, the series diverges.

b. This series is geometric. Moreover, because the ratio of the terms

$$r = \frac{\pi}{6}$$

is less than 1 in absolute value, you can conclude that the series converges.

c. Because the function

$$f(x) = xe^{-x^2}$$

is easily integrated, you can use the Integral Test to conclude that the series converges.

d. The nth term of this series can be compared to the nth term of the harmonic series. After using the Limit Comparison Test, you can conclude that the series diverges.

e. This is an alternating series whose nth term approaches 0. Because $a_{n+1} \le a_n$, you can use the Alternating Series Test to conclude that the series converges.

f. The nth term of this series involves a factorial, which indicates that the Ratio Test may work well. After applying the Ratio Test, you can conclude that the series diverges.

g. The nth term of this series involves a variable that is raised to the nth power, which indicates that the Root Test may work well. After applying the Root Test, you can conclude that the series converges.

SUMMARY OF TESTS FOR SERIES

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$0 < \lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1}a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$ or $= \infty$	Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$ or $= \infty$	Test is inconclusive when $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

9.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Verifying a Formula In Exercises 1–4, verify the formula.

1. $\dfrac{(n+1)!}{(n-2)!} = (n+1)(n)(n-1)$

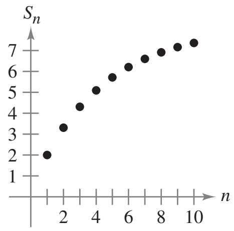
2. $\dfrac{(2k-2)!}{(2k)!} = \dfrac{1}{(2k)(2k-1)}$

3. $1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2k-1) = \dfrac{(2k)!}{2^k k!}$

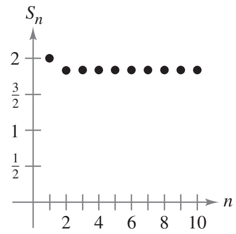
4. $\dfrac{1}{1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2k-5)} = \dfrac{2^k k!(2k-3)(2k-1)}{(2k)!}, \quad k \geq 3$

Matching In Exercises 5–10, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)



(b)



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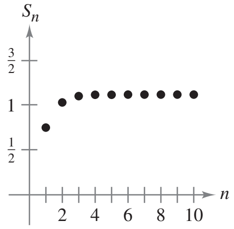

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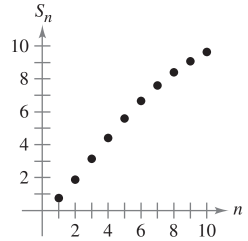

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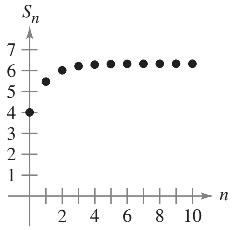
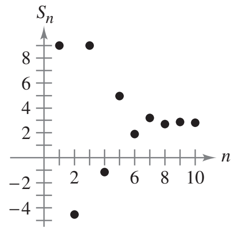


(f)



5. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n\left(\frac{1}{n!}\right)$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!}$

9. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n$

10. $\displaystyle\sum_{n=0}^{\infty} 4e^{-n}$

Numerical, Graphical, and Analytic Analysis In Exercises 11 and 12, (a) verify that the series converges, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums, (d) use the table to estimate the sum of the series, and (e) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	15	20	25
S_n					

11. $\displaystyle\sum_{n=1}^{\infty} n^3\left(\frac{1}{2}\right)^n$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n!}$

Using the Ratio Test In Exercises 13–34, use the Ratio Test to determine the convergence or divergence of the series.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

16. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

17. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

18. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

31. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{2^n+1}$

32. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2n+1)}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n[2 \cdot 4 \cdot 6 \cdot \cdot \cdot (2n)]}{2 \cdot 5 \cdot 8 \cdot \cdot \cdot (3n-1)}$

Using the Root Test **In Exercises 35–50, use the Root Test to determine the convergence or divergence of the series.**

35. $\sum_{n=1}^{\infty} \frac{1}{5^n}$

36. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

37. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

38. $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

39. $\sum_{n=1}^{\infty} \left(\frac{3n+2}{n+3}\right)^n$

40. $\sum_{n=1}^{\infty} \left(\frac{n-2}{5n+1}\right)^n$

41. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

42. $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

43. $\sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$

44. $\sum_{n=0}^{\infty} e^{-3n}$

45. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

46. $\sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$

47. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

48. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

49. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

50. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

Determining Convergence or Divergence **In Exercises 51–68, determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.**

51. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}5}{n}$

52. $\sum_{n=1}^{\infty} \frac{100}{n}$

53. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

54. $\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$

55. $\sum_{n=1}^{\infty} \frac{5n}{2n-1}$

56. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

57. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n}$

58. $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

59. $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

60. $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

61. $\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$

62. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

63. $\sum_{n=1}^{\infty} \frac{n!}{n7^n}$

64. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

65. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$

66. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

67. $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

68. $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n(2n-1)n!}$

Identifying Series **In Exercises 69–72, identify the two series that are the same.**

69. (a) $\sum_{n=1}^{\infty} \frac{n5^n}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{n5^n}{(n+1)!}$

(c) $\sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!}$

70. (a) $\sum_{n=4}^{\infty} n\left(\frac{3}{4}\right)^n$

(b) $\sum_{n=0}^{\infty} (n+1)\left(\frac{3}{4}\right)^n$

(c) $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^{n-1}$

71. (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!}$

72. (a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)2^n}$

Writing an Equivalent Series **In Exercises 73 and 74, write an equivalent series with the index of summation beginning at $n = 0$.**

73. $\sum_{n=1}^{\infty} \frac{n}{7^n}$

74. $\sum_{n=2}^{\infty} \frac{9^n}{(n-2)!}$

Finding the Number of Terms **In Exercises 75 and 76, (a) determine the number of terms required to approximate the sum of the series with an error less than 0.0001, and (b) use a graphing utility to approximate the sum of the series with an error less than 0.0001.**

75. $\sum_{k=1}^{\infty} \frac{(-3)^k}{2^k k!}$

76. $\sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$

Using a Recursively Defined Series **In Exercises 77–82, the terms of a series $\sum_{n=1}^{\infty} a_n$ are defined recursively. Determine the convergence or divergence of the series. Explain your reasoning.**

77. $a_1 = \frac{1}{2},\ a_{n+1} = \frac{4n-1}{3n+2}a_n$

78. $a_1 = 2,\ a_{n+1} = \frac{2n+1}{5n-4}a_n$

79. $a_1 = 1,\ a_{n+1} = \frac{\sin n + 1}{\sqrt{n}}a_n$

80. $a_1 = \frac{1}{5},\ a_{n+1} = \frac{\cos n + 1}{n}a_n$

81. $a_1 = \frac{1}{3},\ a_{n+1} = \left(1 + \frac{1}{n}\right)a_n$

82. $a_1 = \frac{1}{4},\ a_{n+1} = \sqrt[n]{a_n}$

Using the Ratio Test or Root Test **In Exercises 83–86, use the Ratio Test or the Root Test to determine the convergence or divergence of the series.**

83. $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$

84. $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \frac{5}{3^4} + \frac{6}{3^5} + \cdots$

85. $\frac{1}{(\ln 3)^3} + \frac{1}{(\ln 4)^4} + \frac{1}{(\ln 5)^5} + \frac{1}{(\ln 6)^6} + \cdots$

86. $1 + \frac{1 \cdot 3}{1 \cdot 2 \cdot 3} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \cdots$

Finding Values **In Exercises 87–92, find the values of x for which the series converges.**

87. $\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$

88. $\sum_{n=0}^{\infty} \left(\frac{x-3}{5}\right)^n$

89. $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n}$

90. $\sum_{n=0}^{\infty} 3(x-4)^n$

91. $\sum_{n=0}^{\infty} n!\left(\frac{x}{2}\right)^n$

92. $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$

WRITING ABOUT CONCEPTS

93. **Ratio Test** State the Ratio Test.

94. **Root Test** State the Root Test.

95. **Think About It** You are told that the terms of a positive series appear to approach zero rapidly as n approaches infinity. In fact, $a_7 \le 0.0001$. Given no other information, does this imply that the series converges? Support your conclusion with examples.

96. **Think About It** What can you conclude about the convergence or divergence of $\Sigma\, a_n$ for each of the following conditions? Explain your reasoning.

(a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$

(b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

(c) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{3}{2}$

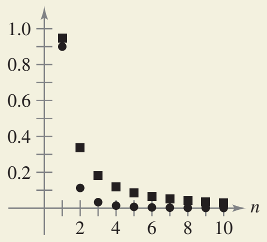
(d) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2$

(e) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

(f) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = e$

97. **Using an Alternating Series** Using the Ratio Test, it is determined that an alternating series converges. Does the series converge conditionally or absolutely? Explain.

98. **HOW DO YOU SEE IT?** The figure shows the first 10 terms of the convergent series $\sum_{n=1}^{\infty} a_n$ and the first 10 terms of the convergent series $\sum_{n=1}^{\infty} \sqrt{a_n}$. Identify the two series and explain your reasoning in making the selection.



99. **Proof** Prove Property 2 of Theorem 9.17.

100. **Proof** Prove Theorem 9.18. (*Hint for Property 1:* If the limit equals $r < 1$, choose a real number R such that $r < R < 1$. By the definitions of the limit, there exists some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for $n > N$.)

Verifying an Inconclusive Test **In Exercises 101–104, verify that the Ratio Test is inconclusive for the p-series.**

101. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

102. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

103. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

104. $\sum_{n=1}^{\infty} \frac{1}{n^p}$

105. **Verifying an Inconclusive Test** Show that the Root Test is inconclusive for the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

106. **Verifying Inconclusive Tests** Show that the Ratio Test and the Root Test are both inconclusive for the logarithmic p-series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

107. **Using Values** Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$$

when (a) $x = 1$, (b) $x = 2$, (c) $x = 3$, and (d) x is a positive integer.

108. **Using a Series** Show that if

$$\sum_{n=1}^{\infty} a_n$$

is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|.$$

PUTNAM EXAM CHALLENGE

109. Show that if the series

$$a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot$$

converges, then the series

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdot \cdot \cdot + \frac{a_n}{n} + \cdot \cdot \cdot$$

converges also.

110. Is the following series convergent or divergent?

$$1 + \frac{1}{2} \cdot \frac{19}{7} + \frac{2!}{3^2}\left(\frac{19}{7}\right)^2 + \frac{3!}{4^3}\left(\frac{19}{7}\right)^3 + \frac{4!}{5^4}\left(\frac{19}{7}\right)^4 + \cdot \cdot \cdot$$

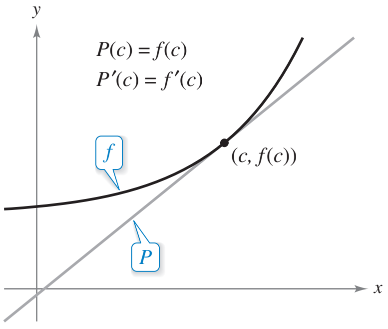
9.7 Taylor Polynomials and Approximations

- Find polynomial approximations of elementary functions and compare them with the elementary functions.
- Find Taylor and Maclaurin polynomial approximations of elementary functions.
- Use the remainder of a Taylor polynomial.

Polynomial Approximations of Elementary Functions

The goal of this section is to show how polynomial functions can be used as approximations for other elementary functions. To find a polynomial function P that approximates another function f, begin by choosing a number c in the domain of f at which f and P have the same value. That is,

$$P(c) = f(c).$$ Graphs of f and P pass through $(c, f(c))$.



Near $(c, f(c))$, the graph of P can be used to approximate the graph of f.
Figure 9.10

The approximating polynomial is said to be **expanded about c** or **centered at c.** Geometrically, the requirement that $P(c) = f(c)$ means that the graph of P passes through the point $(c, f(c))$. Of course, there are many polynomials whose graphs pass through the point $(c, f(c))$. Your task is to find a polynomial whose graph resembles the graph of f near this point. One way to do this is to impose the additional requirement that the slope of the polynomial function be the same as the slope of the graph of f at the point $(c, f(c))$.

$$P'(c) = f'(c)$$ Graphs of f and P have the same slope at $(c, f(c))$.

With these two requirements, you can obtain a simple linear approximation of f, as shown in Figure 9.10.

REMARK Example 1 is not the first time you have used a linear function to approximate another function. The same procedure was used as the basis for Newton's Method.

EXAMPLE 1 First-Degree Polynomial Approximation of $f(x) = e^x$

For the function $f(x) = e^x$, find a first-degree polynomial function $P_1(x) = a_0 + a_1x$ whose value and slope agree with the value and slope of f at $x = 0$.

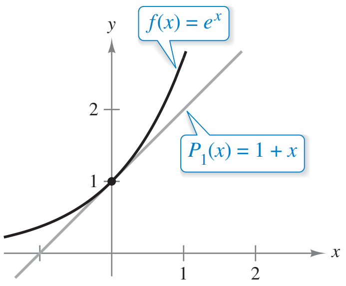
Solution Because $f(x) = e^x$ and $f'(x) = e^x$, the value and the slope of f at $x = 0$ are

$$f(0) = e^0 = 1$$ Value of f at $x = 0$

and

$$f'(0) = e^0 = 1.$$ Slope of f at $x = 0$

Because $P_1(x) = a_0 + a_1x$, you can use the condition that $P_1(0) = f(0)$ to conclude that $a_0 = 1$. Moreover, because $P_1'(x) = a_1$, you can use the condition that $P_1'(0) = f'(0)$ to conclude that $a_1 = 1$. Therefore, $P_1(x) = 1 + x$. Figure 9.11 shows the graphs of $P_1(x) = 1 + x$ and $f(x) = e^x$.



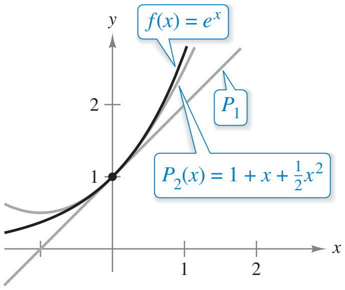
P_1 is the first-degree polynomial approximation of $f(x) = e^x$.
Figure 9.11

In Figure 9.12, you can see that, at points near $(0, 1)$, the graph of the first-degree polynomial function

$$P_1(x) = 1 + x \qquad \text{1st-degree approximation}$$

is reasonably close to the graph of $f(x) = e^x$. As you move away from $(0, 1)$, however, the graphs move farther and farther from each other and the accuracy of the approximation decreases. To improve the approximation, you can impose yet another requirement—that the values of the second derivatives of P and f agree when $x = 0$. The polynomial, P_2, of least degree that satisfies all three requirements $P_2(0) = f(0)$, $P_2{}'(0) = f'(0)$, and $P_2{}''(0) = f''(0)$ can be shown to be

$$P_2(x) = 1 + x + \frac{1}{2}x^2. \qquad \text{2nd-degree approximation}$$



P_2 is the second-degree polynomial approximation of $f(x) = e^x$.
Figure 9.12

Moreover, in Figure 9.12, you can see that P_2 is a better approximation of f than P_1. By requiring that the values of $P_n(x)$ and its first n derivatives match those of $f(x) = e^x$ at $x = 0$, you obtain the nth-degree approximation shown below.

$$P_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdot \cdot \cdot + \frac{1}{n!}x^n \qquad n\text{th-degree approximation}$$
$$\approx e^x$$

EXAMPLE 2 Third-Degree Polynomial Approximation of $f(x) = e^x$

Construct a table comparing the values of the polynomial

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 \qquad \text{3rd-degree approximation}$$

with $f(x) = e^x$ for several values of x near 0.

Solution Using a calculator or a computer, you can obtain the results shown in the table. Note that for $x = 0$, the two functions have the same value, but that as x moves farther away from 0, the accuracy of the approximating polynomial $P_3(x)$ decreases.

x	-1.0	-0.2	-0.1	0	0.1	0.2	1.0
e^x	0.3679	0.81873	0.904837	1	1.105171	1.22140	2.7183
$P_3(x)$	0.3333	0.81867	0.904833	1	1.105167	1.22133	2.6667

TECHNOLOGY A graphing utility can be used to compare the graph of the approximating polynomial with the graph of the function f. For instance, in Figure 9.13, the graph of

$$P_3(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 \qquad \text{3rd-degree approximation}$$

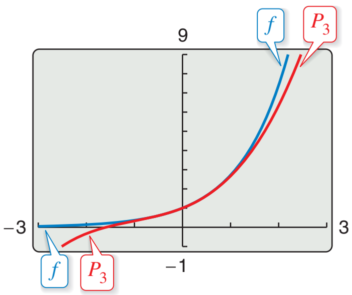
is compared with the graph of $f(x) = e^x$. If you have access to a graphing utility, try comparing the graphs of

$$P_4(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 \qquad \text{4th-degree approximation}$$
$$P_5(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5 \qquad \text{5th-degree approximation}$$

and

$$P_6(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5 + \tfrac{1}{720}x^6 \qquad \text{6th-degree approximation}$$

with the graph of f. What do you notice?



P_3 is the third-degree polynomial approximation of $f(x) = e^x$.
Figure 9.13

BROOK TAYLOR (1685–1731)

Although Taylor was not the first to seek polynomial approximations of transcendental functions, his account published in 1715 was one of the first comprehensive works on the subject.

See LarsonCalculus.com to read more of this biography.

Taylor and Maclaurin Polynomials

The polynomial approximation of

$$f(x) = e^x$$

in Example 2 is expanded about $c = 0$. For expansions about an arbitrary value of c, it is convenient to write the polynomial in the form

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots + a_n(x - c)^n.$$

In this form, repeated differentiation produces

$$\begin{aligned} P_n'(x) &= a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \cdots + na_n(x - c)^{n-1} \\ P_n''(x) &= 2a_2 + 2(3a_3)(x - c) + \cdots + n(n - 1)a_n(x - c)^{n-2} \\ P_n'''(x) &= 2(3a_3) + \cdots + n(n - 1)(n - 2)a_n(x - c)^{n-3} \\ &\vdots \\ P_n^{(n)}(x) &= n(n - 1)(n - 2) \cdots (2)(1)a_n. \end{aligned}$$

Letting $x = c$, you then obtain

$$P_n(c) = a_0, \quad P_n'(c) = a_1, \quad P_n''(c) = 2a_2, \ldots, \quad P_n^{(n)}(c) = n!a_n$$

and because the values of f and its first n derivatives must agree with the values of P_n and its first n derivatives at $x = c$, it follows that

$$f(c) = a_0, \quad f'(c) = a_1, \quad \frac{f''(c)}{2!} = a_2, \quad \ldots, \quad \frac{f^{(n)}(c)}{n!} = a_n.$$

With these coefficients, you can obtain the following definition of **Taylor polynomials,** named after the English mathematician Brook Taylor, and **Maclaurin polynomials,** named after the English mathematician Colin Maclaurin (1698–1746).

REMARK Maclaurin polynomials are special types of Taylor polynomials for which $c = 0$.

Definitions of *n*th Taylor Polynomial and *n*th Maclaurin Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the ***n*th Taylor polynomial for *f* at *c*.** If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the ***n*th Maclaurin polynomial for *f*.**

EXAMPLE 3 A Maclaurin Polynomial for $f(x) = e^x$

Find the nth Maclaurin polynomial for

$$f(x) = e^x.$$

Solution From the discussion on the preceding page, the nth Maclaurin polynomial for

$$f(x) = e^x$$

is

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

FOR FURTHER INFORMATION To see how to use series to obtain other approximations to e, see the article "Novel Series-based Approximations to e" by John Knox and Harlan J. Brothers in *The College Mathematics Journal.* To view this article, go to *MathArticles.com.*

The Granger Collection

EXAMPLE 4 Finding Taylor Polynomials for $\ln x$

Find the Taylor polynomials P_0, P_1, P_2, P_3, and P_4 for

$$f(x) = \ln x$$

centered at $c = 1$.

Solution Expanding about $c = 1$ yields the following.

$$\begin{aligned}
f(x) &= \ln x & f(1) &= \ln 1 = 0\\
f'(x) &= \frac{1}{x} & f'(1) &= \frac{1}{1} = 1\\
f''(x) &= -\frac{1}{x^2} & f''(1) &= -\frac{1}{1^2} = -1\\
f'''(x) &= \frac{2!}{x^3} & f'''(1) &= \frac{2!}{1^3} = 2\\
f^{(4)}(x) &= -\frac{3!}{x^4} & f^{(4)}(1) &= -\frac{3!}{1^4} = -6
\end{aligned}$$

Therefore, the Taylor polynomials are as follows.

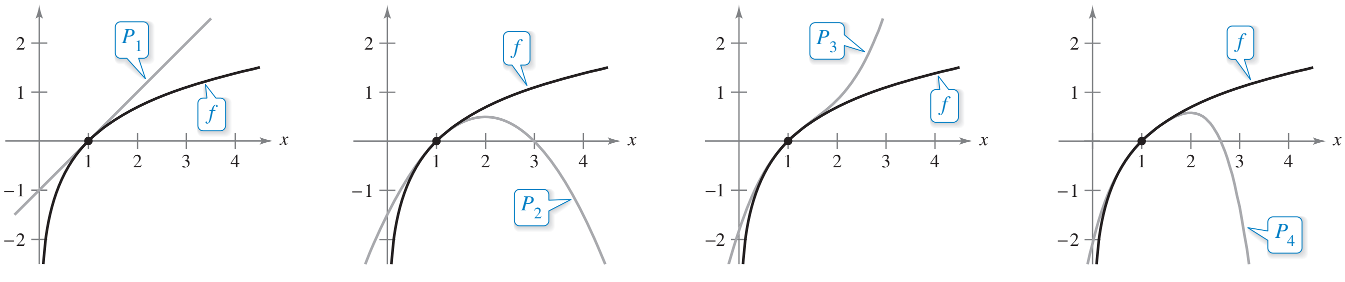
$$\begin{aligned}
P_0(x) &= f(1) = 0\\
P_1(x) &= f(1) + f'(1)(x-1) = (x-1)\\
P_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2\\
&= (x-1) - \frac{1}{2}(x-1)^2\\
P_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3\\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3\\
P_4(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4\\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4
\end{aligned}$$

Figure 9.14 compares the graphs of P_1, P_2, P_3, and P_4 with the graph of $f(x) = \ln x$. Note that near $x = 1$, the graphs are nearly indistinguishable. For instance,

$$P_4(1.1) \approx 0.0953083$$

and

$$\ln(1.1) \approx 0.0953102.$$



As n increases, the graph of P_n becomes a better and better approximation of the graph of $f(x) = \ln x$ near $x = 1$.
Figure 9.14

EXAMPLE 5 Finding Maclaurin Polynomials for cos *x*

Find the Maclaurin polynomials P_0, P_2, P_4, and P_6 for $f(x) = \cos x$. Use $P_6(x)$ to approximate the value of $\cos(0.1)$.

Solution Expanding about $c = 0$ yields the following.

$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -\cos 0 = -1 \\ f'''(x) &= \sin x & f'''(0) &= \sin 0 = 0 \end{aligned}$$

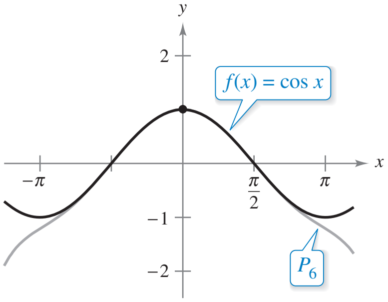
Through repeated differentiation, you can see that the pattern 1, 0, -1, 0 continues, and you obtain the Maclaurin polynomials

$$P_0(x) = 1, \quad P_2(x) = 1 - \frac{1}{2!}x^2, \quad P_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4,$$

and

$$P_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6.$$

Using $P_6(x)$, you obtain the approximation $\cos(0.1) \approx 0.995004165$, which coincides with the calculator value to nine decimal places. Figure 9.15 compares the graphs of $f(x) = \cos x$ and P_6.



Near (0, 1), the graph of P_6 can be used to approximate the graph of $f(x) = \cos x$.
Figure 9.15

Note in Example 5 that the Maclaurin polynomials for $\cos x$ have only even powers of x. Similarly, the Maclaurin polynomials for $\sin x$ have only odd powers of x (see Exercise 17). This is not generally true of the Taylor polynomials for $\sin x$ and $\cos x$ expanded about $c \neq 0$, as you can see in the next example.

EXAMPLE 6 Finding a Taylor Polynomial for sin *x*

See LarsonCalculus.com for an interactive version of this type of example.

Find the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$.

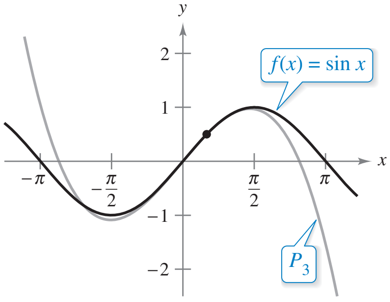
Solution Expanding about $c = \pi/6$ yields the following.

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{6}\right) &= \sin\frac{\pi}{6} = \frac{1}{2} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{6}\right) &= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{6}\right) &= -\sin\frac{\pi}{6} = -\frac{1}{2} \\ f'''(x) &= -\cos x & f'''\left(\frac{\pi}{6}\right) &= -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2} \end{aligned}$$

So, the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$, is

$$\begin{aligned} P_3(x) &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x - \frac{\pi}{6}\right)^3 \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3. \end{aligned}$$

Figure 9.16 compares the graphs of $f(x) = \sin x$ and P_3.



Near $(\pi/6, 1/2)$, the graph of P_3 can be used to approximate the graph of $f(x) = \sin x$.
Figure 9.16

Taylor polynomials and Maclaurin polynomials can be used to approximate the value of a function at a specific point. For instance, to approximate the value of $\ln(1.1)$, you can use Taylor polynomials for $f(x) = \ln x$ expanded about $c = 1$, as shown in Example 4, or you can use Maclaurin polynomials, as shown in Example 7.

EXAMPLE 7 Approximation Using Maclaurin Polynomials

Use a fourth Maclaurin polynomial to approximate the value of $\ln(1.1)$.

Solution Because 1.1 is closer to 1 than to 0, you should consider Maclaurin polynomials for the function $g(x) = \ln(1 + x)$.

$$\begin{aligned} g(x) &= \ln(1+x) & g(0) &= \ln(1+0) = 0 \\ g'(x) &= (1+x)^{-1} & g'(0) &= (1+0)^{-1} = 1 \\ g''(x) &= -(1+x)^{-2} & g''(0) &= -(1+0)^{-2} = -1 \\ g'''(x) &= 2(1+x)^{-3} & g'''(0) &= 2(1+0)^{-3} = 2 \\ g^{(4)}(x) &= -6(1+x)^{-4} & g^{(4)}(0) &= -6(1+0)^{-4} = -6 \end{aligned}$$

Note that you obtain the same coefficients as in Example 4. Therefore, the fourth Maclaurin polynomial for $g(x) = \ln(1 + x)$ is

$$\begin{aligned} P_4(x) &= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4. \end{aligned}$$

Consequently,

$$\ln(1.1) = \ln(1 + 0.1) \approx P_4(0.1) \approx 0.0953083.$$

Exploration

Check to see that the fourth Taylor polynomial (from Example 4), evaluated at $x = 1.1$, yields the same result as the fourth Maclaurin polynomial in Example 7.

The table below illustrates the accuracy of the Maclaurin polynomial approximation of the calculator value of $\ln(1.1)$. You can see that as n increases, $P_n(0.1)$ approaches the calculator value of 0.0953102.

Maclaurin Polynomials and Approximations of $\ln(1 + x)$ at $x = 0.1$

n	1	2	3	4
$P_n(0.1)$	0.1000000	0.0950000	0.0953333	0.0953083

On the other hand, the table below illustrates that as you move away from the expansion point $c = 0$, the accuracy of the approximation decreases.

Fourth Maclaurin Polynomial Approximation of $\ln(1 + x)$

x	0	0.1	0.5	0.75	1.0
$\ln(1 + x)$	0	0.0953102	0.4054651	0.5596158	0.6931472
$P_4(x)$	0	0.0953083	0.4010417	0.5302734	0.5833333

These two tables illustrate two very important points about the accuracy of Taylor (or Maclaurin) polynomials for use in approximations.

1. The approximation is usually better for higher-degree Taylor (or Maclaurin) polynomials than for those of lower degree.
2. The approximation is usually better at x-values close to c than at x-values far from c.

Remainder of a Taylor Polynomial

An approximation technique is of little value without some idea of its accuracy. To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, you can use the concept of a **remainder** $R_n(x)$, defined as follows.

$$\underbrace{f(x)}_{\text{Exact value}} = \underbrace{P_n(x)}_{\text{Approximate value}} + \underbrace{R_n(x)}_{\text{Remainder}}$$

So, $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the **error** associated with the approximation. That is,

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|.$$

The next theorem gives a general procedure for estimating the remainder associated with a Taylor polynomial. This important theorem is called **Taylor's Theorem,** and the remainder given in the theorem is called the **Lagrange form of the remainder.**

THEOREM 9.19 Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

One useful consequence of Taylor's Theorem is that

$$|R_n(x)| \le \frac{|x - c|^{n+1}}{(n + 1)!} \max |f^{(n+1)}(z)|$$

where $\max|f^{(n+1)}(z)|$ is the maximum value of $f^{(n+1)}(z)$ between x and c.

For $n = 0$, Taylor's Theorem states that if f is differentiable in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(z)(x - c) \quad \text{or} \quad f'(z) = \frac{f(x) - f(c)}{x - c}.$$

Do you recognize this special case of Taylor's Theorem? (It is the Mean Value Theorem.)

When applying Taylor's Theorem, you should not expect to be able to find the exact value of z. (If you could do this, an approximation would not be necessary.) Rather, you are trying to find bounds for $f^{(n+1)}(z)$ from which you are able to tell how large the remainder $R_n(x)$ is.

EXAMPLE 8 Determining the Accuracy of an Approximation

The third Maclaurin polynomial for $\sin x$ is

$$P_3(x) = x - \frac{x^3}{3!}.$$

Use Taylor's Theorem to approximate $\sin(0.1)$ by $P_3(0.1)$ and determine the accuracy of the approximation.

Solution Using Taylor's Theorem, you have

$$\sin x = x - \frac{x^3}{3!} + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!}x^4$$

where $0 < z < 0.1$. Therefore,

$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.1 - 0.000167 = 0.099833.$$

Because $f^{(4)}(z) = \sin z$, it follows that the error $|R_3(0.1)|$ can be bounded as follows.

$$0 < R_3(0.1) = \frac{\sin z}{4!}(0.1)^4 < \frac{0.0001}{4!} \approx 0.000004$$

This implies that

$$0.099833 < \sin(0.1) \approx 0.099833 + R_3(0.1) < 0.099833 + 0.000004$$

or

$$0.099833 < \sin(0.1) < 0.099837.$$

REMARK Note that when you use a calculator,

$$\sin(0.1) \approx 0.0998334.$$

EXAMPLE 9 Approximating a Value to a Desired Accuracy

Determine the degree of the Taylor polynomial $P_n(x)$ expanded about $c = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than 0.001.

Solution Following the pattern of Example 4, you can see that the $(n + 1)$st derivative of $f(x) = \ln x$ is

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}.$$

Using Taylor's Theorem, you know that the error $|R_n(1.2)|$ is

$$\begin{aligned}|R_n(1.2)| &= \left|\frac{f^{(n+1)}(z)}{(n+1)!}(1.2 - 1)^{n+1}\right| \\ &= \frac{n!}{z^{n+1}}\left[\frac{1}{(n+1)!}\right](0.2)^{n+1} \\ &= \frac{(0.2)^{n+1}}{z^{n+1}(n+1)}\end{aligned}$$

where $1 < z < 1.2$. In this interval, $(0.2)^{n+1}/[z^{n+1}(n + 1)]$ is less than $(0.2)^{n+1}/(n + 1)$. So, you are seeking a value of n such that

$$\frac{(0.2)^{n+1}}{(n+1)} < 0.001 \quad \Longrightarrow \quad 1000 < (n+1)5^{n+1}.$$

By trial and error, you can determine that the least value of n that satisfies this inequality is $n = 3$. So, you would need the third Taylor polynomial to achieve the desired accuracy in approximating $\ln(1.2)$.

REMARK Note that when you use a calculator,

$$P_3(1.2) \approx 0.1827$$

and

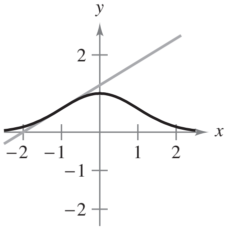
$$\ln(1.2) \approx 0.1823.$$

9.7 Exercises

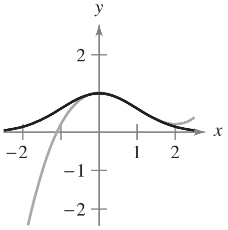
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching **In Exercises 1–4, match the Taylor polynomial approximation of the function $f(x) = e^{-x^2/2}$ with the corresponding graph. [The graphs are labeled (a), (b), (c), and (d).]**

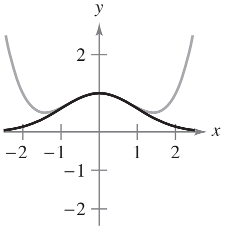
(a)



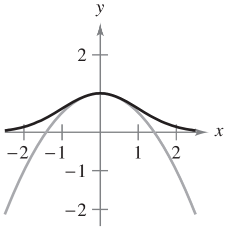
(b)



(c)



(d)



1. $g(x) = -\frac{1}{2}x^2 + 1$

2. $g(x) = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

3. $g(x) = e^{-1/2}[(x + 1) + 1]$

4. $g(x) = e^{-1/2}\left[\frac{1}{3}(x - 1)^3 - (x - 1) + 1\right]$

Finding a First-Degree Polynomial Approximation **In Exercises 5–8, find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of f at $x = c$. Use a graphing utility to graph f and P_1. What is P_1 called?**

5. $f(x) = \dfrac{\sqrt{x}}{4}, \quad c = 4$

6. $f(x) = \dfrac{6}{\sqrt[3]{x}}, \quad c = 8$

7. $f(x) = \sec x, \quad c = \dfrac{\pi}{4}$

8. $f(x) = \tan x, \quad c = \dfrac{\pi}{4}$

Graphical and Numerical Analysis **In Exercises 9 and 10, use a graphing utility to graph f and its second-degree polynomial approximation P_2 at $x = c$. Complete the table comparing the values of f and P_2.**

9. $f(x) = \dfrac{4}{\sqrt{x}}, \quad c = 1$

$P_2(x) = 4 - 2(x - 1) + \frac{3}{2}(x - 1)^2$

x	0	0.8	0.9	1	1.1	1.2	2
$f(x)$							
$P_2(x)$							

10. $f(x) = \sec x, \quad c = \dfrac{\pi}{4}$

$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \dfrac{\pi}{4}\right) + \dfrac{3}{2}\sqrt{2}\left(x - \dfrac{\pi}{4}\right)^2$

x	-2.15	0.585	0.685	$\frac{\pi}{4}$	0.885	0.985	1.785
$f(x)$							
$P_2(x)$							

11. Conjecture Consider the function $f(x) = \cos x$ and its Maclaurin polynomials P_2, P_4, and P_6 (see Example 5).

(a) Use a graphing utility to graph f and the indicated polynomial approximations.

(b) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2$, 4, and 6.

(c) Use the results in part (b) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

12. Conjecture Consider the function $f(x) = x^2e^x$.

(a) Find the Maclaurin polynomials P_2, P_3, and P_4 for f.

(b) Use a graphing utility to graph f, P_2, P_3, and P_4.

(c) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2$, 3, and 4.

(d) Use the results in part (c) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

Finding a Maclaurin Polynomial **In Exercises 13–24, find the nth Maclaurin polynomial for the function.**

13. $f(x) = e^{4x}, \quad n = 4$

14. $f(x) = e^{-x}, \quad n = 5$

15. $f(x) = e^{-x/2}, \quad n = 4$

16. $f(x) = e^{x/3}, \quad n = 4$

17. $f(x) = \sin x, \quad n = 5$

18. $f(x) = \cos \pi x, \quad n = 4$

19. $f(x) = xe^x, \quad n = 4$

20. $f(x) = x^2e^{-x}, \quad n = 4$

21. $f(x) = \dfrac{1}{x + 1}, \quad n = 5$

22. $f(x) = \dfrac{x}{x + 1}, \quad n = 4$

23. $f(x) = \sec x, \quad n = 2$

24. $f(x) = \tan x, \quad n = 3$

Finding a Taylor Polynomial **In Exercises 25–30, find the nth Taylor polynomial centered at c.**

25. $f(x) = \dfrac{2}{x}, \quad n = 3, \quad c = 1$

26. $f(x) = \dfrac{1}{x^2}, \quad n = 4, \quad c = 2$

27. $f(x) = \sqrt{x}, \quad n = 3, \quad c = 4$

28. $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

29. $f(x) = \ln x, \quad n = 4, \quad c = 2$

30. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

Finding Taylor Polynomials Using Technology **In Exercises 31 and 32, use a computer algebra system to find the indicated Taylor polynomials for the function f. Graph the function and the Taylor polynomials.**

31. $f(x) = \tan \pi x$

(a) $n = 3, \quad c = 0$

(b) $n = 3, \quad c = 1/4$

32. $f(x) = \dfrac{1}{x^2 + 1}$

(a) $n = 4, \quad c = 0$

(b) $n = 4, \quad c = 1$

33. Numerical and Graphical Approximations

(a) Use the Maclaurin polynomials $P_1(x)$, $P_3(x)$, and $P_5(x)$ for $f(x) = \sin x$ to complete the table.

x	0	0.25	0.50	0.75	1.00
$\sin x$	0	0.2474	0.4794	0.6816	0.8415
$P_1(x)$					
$P_3(x)$					
$P_5(x)$					

(b) Use a graphing utility to graph $f(x) = \sin x$ and the Maclaurin polynomials in part (a).

(c) Describe the change in accuracy of a polynomial approximation as the distance from the point where the polynomial is centered increases.

34. Numerical and Graphical Approximations

(a) Use the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_4(x)$ for $f(x) = e^x$ centered at $c = 1$ to complete the table.

x	1.00	1.25	1.50	1.75	2.00
e^x	e	3.4903	4.4817	5.7546	7.3891
$P_1(x)$					
$P_2(x)$					
$P_4(x)$					

(b) Use a graphing utility to graph $f(x) = e^x$ and the Taylor polynomials in part (a).

(c) Describe the change in accuracy of polynomial approximations as the degree increases.

Numerical and Graphical Approximations **In Exercises 35 and 36, (a) find the Maclaurin polynomial $P_3(x)$ for $f(x)$, (b) complete the table for $f(x)$ and $P_3(x)$, and (c) sketch the graphs of $f(x)$ and $P_3(x)$ on the same set of coordinate axes.**

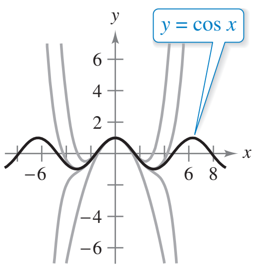
x	−0.75	−0.50	−0.25	0	0.25	0.50	0.75
$f(x)$							
$P_3(x)$							

35. $f(x) = \arcsin x$

36. $f(x) = \arctan x$

Identifying Maclaurin Polynomials **In Exercises 37–40, the graph of $y = f(x)$ is shown with four of its Maclaurin polynomials. Identify the Maclaurin polynomials and use a graphing utility to confirm your results.**

37.



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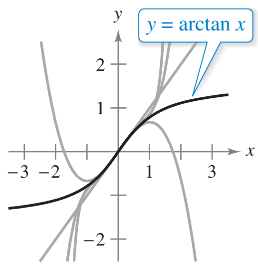

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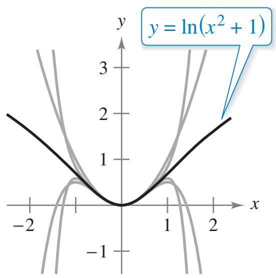

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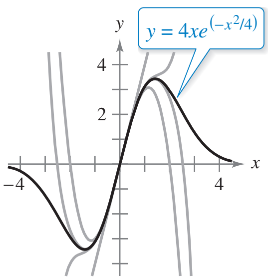


Approximating a Function Value **In Exercises 41–44, approximate the function at the given value of x, using the polynomial found in the indicated exercise.**

41. $f(x) = e^{4x}, \quad f\left(\dfrac{1}{4}\right)$, Exercise 13

42. $f(x) = x^2e^{-x}, \quad f\left(\frac{1}{5}\right)$, Exercise 20

43. $f(x) = \ln x, \quad f(2.1)$, Exercise 29

44. $f(x) = x^2 \cos x, \quad f\left(\dfrac{7\pi}{8}\right)$, Exercise 30

Using Taylor's Theorem **In Exercises 45–48, use Taylor's Theorem to obtain an upper bound for the error of the approximation. Then calculate the exact value of the error.**

45. $\cos(0.3) \approx 1 - \dfrac{(0.3)^2}{2!} + \dfrac{(0.3)^4}{4!}$

46. $e \approx 1 + 1 + \dfrac{1^2}{2!} + \dfrac{1^3}{3!} + \dfrac{1^4}{4!} + \dfrac{1^5}{5!}$

47. $\arcsin(0.4) \approx 0.4 + \dfrac{(0.4)^3}{2 \cdot 3}$

48. $\arctan(0.4) \approx 0.4 - \dfrac{(0.4)^3}{3}$

Finding a Degree **In Exercises 49–52, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.001.**

49. $\sin(0.3)$

50. $\cos(0.1)$

51. $e^{0.6}$

52. $\ln(1.25)$

Finding a Degree Using Technology **In Exercises 53 and 54, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.0001. Use a computer algebra system to obtain and evaluate the required derivative.**

53. $f(x) = \ln(x + 1)$, approximate $f(0.5)$.

54. $f(x) = e^{-\pi x}$, approximate $f(1.3)$.

Finding Values **In Exercises 55–58, determine the values of x for which the function can be replaced by the Taylor polynomial if the error cannot exceed 0.001.**

55. $f(x) = e^x \approx 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}, \quad x < 0$

56. $f(x) = \sin x \approx x - \dfrac{x^3}{3!}$

57. $f(x) = \cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

58. $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \dfrac{4}{3}x^3$

WRITING ABOUT CONCEPTS

59. Polynomial Approximation An elementary function is approximated by a polynomial. In your own words, describe what is meant by saying that the polynomial is *expanded about c* or *centered at c*.

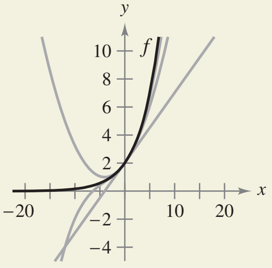
60. Polynomial Approximation When an elementary function f is approximated by a second-degree polynomial P_2 centered at c, what is known about f and P_2 at c? Explain your reasoning.

61. Taylor Polynomial State the definition of an nth-degree Taylor polynomial of f centered at c.

62. Accuracy of a Taylor Polynomial Describe the accuracy of the nth-degree Taylor polynomial of f centered at c as the distance between c and x increases.

63. Accuracy of a Taylor Polynomial In general, how does the accuracy of a Taylor polynomial change as the degree of the polynomial increases? Explain your reasoning.

64. HOW DO YOU SEE IT? The graphs show first-, second-, and third-degree polynomial approximations P_1, P_2, and P_3 of a function f. Label the graphs of P_1, P_2, and P_3. To print an enlarged copy of the graph, go to *MathGraphs.com.*



65. Comparing Maclaurin Polynomials

(a) Compare the Maclaurin polynomials of degree 4 and degree 5, respectively, for the functions $f(x) = e^x$ and $g(x) = xe^x$. What is the relationship between them?

(b) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 6 for the function $g(x) = x \sin x$.

(c) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 4 for the function $g(x) = (\sin x)/x$.

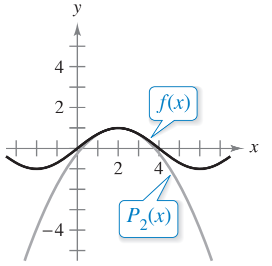
66. Differentiating Maclaurin Polynomials

(a) Differentiate the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ and compare the result with the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) Differentiate the Maclaurin polynomial of degree 6 for $f(x) = \cos x$ and compare the result with the Maclaurin polynomial of degree 5 for $g(x) = \sin x$.

(c) Differentiate the Maclaurin polynomial of degree 4 for $f(x) = e^x$. Describe the relationship between the two series.

67. Graphical Reasoning The figure shows the graphs of the function $f(x) = \sin(\pi x/4)$ and the second-degree Taylor polynomial $P_2(x) = 1 - (\pi^2/32)(x - 2)^2$ centered at $x = 2$.



(a) Use the symmetry of the graph of f to write the second-degree Taylor polynomial $Q_2(x)$ for f centered at $x = -2$.

(b) Use a horizontal translation of the result in part (a) to find the second-degree Taylor polynomial $R_2(x)$ for f centered at $x = 6$.

(c) Is it possible to use a horizontal translation of the result in part (a) to write a second-degree Taylor polynomial for f centered at $x = 4$? Explain.

68. Proof Prove that if f is an odd function, then its nth Maclaurin polynomial contains only terms with odd powers of x.

69. Proof Prove that if f is an even function, then its nth Maclaurin polynomial contains only terms with even powers of x.

70. Proof Let $P_n(x)$ be the nth Taylor polynomial for f at c. Prove that $P_n(c) = f(c)$ and $P^{(k)}(c) = f^{(k)}(c)$ for $1 \le k \le n$. (See Exercises 9 and 10.)

71. Writing The proof in Exercise 70 guarantees that the Taylor polynomial and its derivatives agree with the function and its derivatives at $x = c$. Use the graphs and tables in Exercises 33–36 to discuss what happens to the accuracy of the Taylor polynomial as you move away from $x = c$.

9.8 Power Series

- Understand the definition of a power series.
- Find the radius and interval of convergence of a power series.
- Determine the endpoint convergence of a power series.
- Differentiate and integrate a power series.

Power Series

In Section 9.7, you were introduced to the concept of approximating functions by Taylor polynomials. For instance, the function $f(x) = e^x$ can be *approximated* by its third-degree Maclaurin polynomial

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

In that section, you saw that the higher the degree of the approximating polynomial, the better the approximation becomes.

In this and the next two sections, you will see that several important types of functions, including $f(x) = e^x$, can be represented *exactly* by an infinite series called a **power series.** For example, the power series representation for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

For each real number x, it can be shown that the infinite series on the right converges to the number e^x. Before doing this, however, some preliminary results dealing with power series will be discussed—beginning with the next definition.

Exploration

Graphical Reasoning
Use a graphing utility to approximate the graph of each power series near $x = 0$. (Use the first several terms of each series.) Each series represents a well-known function. What is the function?

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

e. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at** c**,** where c is a constant.

REMARK To simplify the notation for power series, assume that $(x - c)^0 = 1$, even when $x = c$.

EXAMPLE 1 Power Series

a. The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

b. The following power series is centered at -1.

$$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \cdots$$

c. The following power series is centered at 1.

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

Radius and Interval of Convergence

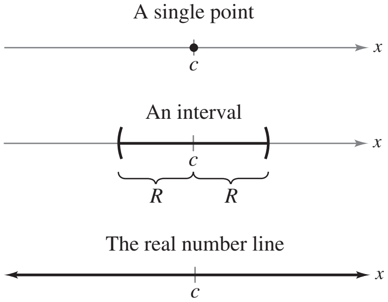
A power series in x can be viewed as a function of x

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

where the *domain of f* is the set of all x for which the power series converges. Determination of the domain of a power series is the primary concern in this section. Of course, every power series converges at its center c because

$$\begin{aligned} f(c) &= \sum_{n=0}^{\infty} a_n(c-c)^n \\ &= a_0(1) + 0 + 0 + \cdots + 0 + \cdots \\ &= a_0. \end{aligned}$$

So, c always lies in the domain of f. Theorem 9.20 (see below) states that the domain of a power series can take three basic forms: a single point, an interval centered at c, or the entire real number line, as shown in Figure 9.17.



The domain of a power series has only three basic forms: a single point, an interval centered at c, or the entire real number line.
Figure 9.17

THEOREM 9.20 Convergence of a Power Series

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for

$$|x-c| < R$$

and diverges for

$$|x-c| > R.$$

3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, then the radius of convergence is $R = 0$. If the series converges for all x, then the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

To determine the radius of convergence of a power series, use the Ratio Test, as demonstrated in Examples 2, 3, and 4.

EXAMPLE 2 Finding the Radius of Convergence

Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$.

Solution For $x = 0$, you obtain

$$f(0) = \sum_{n=0}^{\infty} n!0^n = 1 + 0 + 0 + \cdots = 1.$$

For any fixed value of x such that $|x| > 0$, let $u_n = n!x^n$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= |x| \lim_{n\to\infty}(n+1) \\ &= \infty.\end{aligned}$$

Therefore, by the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0. So, the radius of convergence is $R = 0$.

EXAMPLE 3 Finding the Radius of Convergence

Find the radius of convergence of

$$\sum_{n=0}^{\infty} 3(x-2)^n.$$

Solution For $x \neq 2$, let $u_n = 3(x-2)^n$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right| \\ &= \lim_{n\to\infty}|x-2| \\ &= |x-2|.\end{aligned}$$

By the Ratio Test, the series converges for $|x-2| < 1$ and diverges for $|x-2| > 1$. Therefore, the radius of convergence of the series is $R = 1$.

EXAMPLE 4 Finding the Radius of Convergence

Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

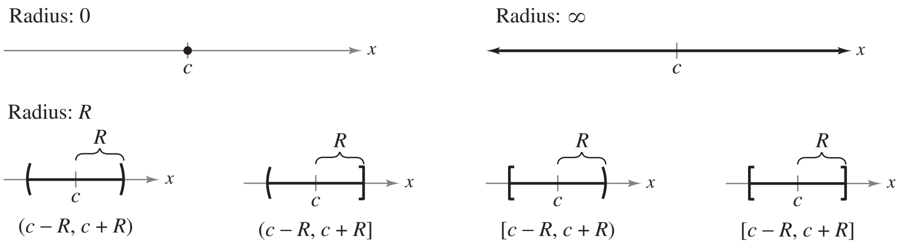
Solution Let $u_n = (-1)^n x^{2n+1}/(2n+1)!$. Then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}}{\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| \\ &= \lim_{n\to\infty}\frac{x^2}{(2n+3)(2n+2)}.\end{aligned}$$

For any *fixed* value of x, this limit is 0. So, by the Ratio Test, the series converges for all x. Therefore, the radius of convergence is $R = \infty$.

Endpoint Convergence

Note that for a power series whose radius of convergence is a finite number R, Theorem 9.20 says nothing about the convergence at the *endpoints* of the interval of convergence. Each endpoint must be tested separately for convergence or divergence. As a result, the interval of convergence of a power series can take any one of the six forms shown in Figure 9.18.



Intervals of convergence
Figure 9.18

EXAMPLE 5 Finding the Interval of Convergence

See LarsonCalculus.com for an interactive version of this type of example.

Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Solution Letting $u_n = x^n/n$ produces

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)}}{\frac{x^n}{n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{nx}{n+1}\right| \\
&= |x|.
\end{aligned}$$

So, by the Ratio Test, the radius of convergence is $R = 1$. Moreover, because the series is centered at 0, it converges in the interval $(-1, 1)$. This interval, however, is not necessarily the *interval of convergence.* To determine this, you must test for convergence at each endpoint. When $x = 1$, you obtain the *divergent* harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots. \qquad \text{Diverges when } x = 1.$$

When $x = -1$, you obtain the *convergent* alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots. \qquad \text{Converges when } x = -1.$$

So, the interval of convergence for the series is $[-1, 1)$, as shown in Figure 9.19.

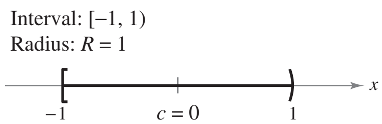


Figure 9.19

EXAMPLE 6 Finding the Interval of Convergence

Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{2^n}$.

Solution Letting $u_n = (-1)^n(x+1)^n/2^n$ produces

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}}}{\dfrac{(-1)^n(x+1)^n}{2^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{2^n(x+1)}{2^{n+1}}\right| \\
&= \left|\frac{x+1}{2}\right|.
\end{aligned}$$

By the Ratio Test, the series converges for

$$\left|\frac{x+1}{2}\right| < 1$$

or $|x+1| < 2$. So, the radius of convergence is $R = 2$. Because the series is centered at $x = -1$, it will converge in the interval $(-3, 1)$. Furthermore, at the endpoints, you have

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$$

Diverges when $x = -3$.

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$$

Diverges when $x = 1$.

both of which diverge. So, the interval of convergence is $(-3, 1)$, as shown in Figure 9.20.

Interval: $(-3, 1)$
Radius: $R = 2$

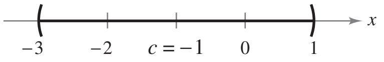


Figure 9.20

EXAMPLE 7 Finding the Interval of Convergence

Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Solution Letting $u_n = x^n/n^2$ produces

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)^2}{x^n/n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2x}{(n+1)^2}\right| \\
&= |x|.
\end{aligned}$$

So, the radius of convergence is $R = 1$. Because the series is centered at $x = 0$, it converges in the interval $(-1, 1)$. When $x = 1$, you obtain the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

Converges when $x = 1$.

When $x = -1$, you obtain the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots.$$

Converges when $x = -1$.

Therefore, the interval of convergence is $[-1, 1]$.

JAMES GREGORY (1638–1675)

One of the earliest mathematicians to work with power series was a Scotsman, James Gregory. He developed a power series method for interpolating table values—a method that was later used by Brook Taylor in the development of Taylor polynomials and Taylor series.

Differentiation and Integration of Power Series

Power series representation of functions has played an important role in the development of calculus. In fact, much of Newton's work with differentiation and integration was done in the context of power series—especially his work with complicated algebraic functions and transcendental functions. Euler, Lagrange, Leibniz, and the Bernoullis all used power series extensively in calculus.

Once you have defined a function with a power series, it is natural to wonder how you can determine the characteristics of the function. Is it continuous? Differentiable? Theorem 9.21, which is stated without proof, answers these questions.

THEOREM 9.21 Properties of Functions Defined by Power Series

If the function

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n(x-c)^n \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots \end{aligned}$$

has a radius of convergence of $R > 0$, then, on the interval

$$(c - R, c + R)$$

f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} na_n(x-c)^{n-1} \\ &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \end{aligned}$$

2. $$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} \\ &= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots \end{aligned}$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence,* however, may differ as a result of the behavior at the endpoints.

Theorem 9.21 states that, in many ways, a function defined by a power series behaves like a polynomial. It is continuous in its interval of convergence, and both its derivative and its antiderivative can be determined by differentiating and integrating each term of the power series. For instance, the derivative of the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

is

$$\begin{aligned} f'(x) &= 1 + (2)\frac{x}{2} + (3)\frac{x^2}{3!} + (4)\frac{x^3}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= f(x). \end{aligned}$$

Notice that $f'(x) = f(x)$. Do you recognize this function?

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EXAMPLE 8 Intervals of Convergence for $f(x)$, $f'(x)$, and $\int f(x)\,dx$

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

Find the interval of convergence for each of the following.

a. $\int f(x)\,dx$ **b.** $f(x)$ **c.** $f'(x)$

Solution By Theorem 9.21, you have

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} x^{n-1} \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

and

$$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} \\ &= C + \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \cdots. \end{aligned}$$

By the Ratio Test, you can show that each series has a radius of convergence of $R = 1$. Considering the interval $(-1, 1)$, you have the following.

a. For $\int f(x)\,dx$, the series

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} \qquad \text{Interval of convergence: } [-1, 1]$$

converges for $x = \pm 1$, and its interval of convergence is $[-1, 1]$. See Figure 9.21(a).

b. For $f(x)$, the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad \text{Interval of convergence: } [-1, 1)$$

converges for $x = -1$ and diverges for $x = 1$. So, its interval of convergence is $[-1, 1)$. See Figure 9.21(b).

c. For $f'(x)$, the series

$$\sum_{n=1}^{\infty} x^{n-1} \qquad \text{Interval of convergence: } (-1, 1)$$

diverges for $x = \pm 1$, and its interval of convergence is $(-1, 1)$. See Figure 9.21(c).

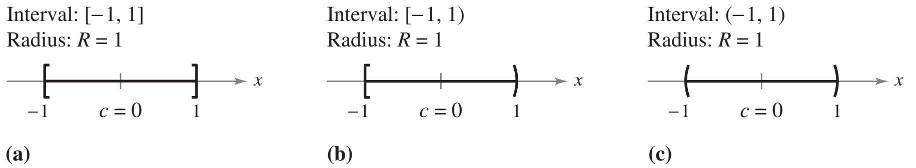


Figure 9.21

From Example 8, it appears that of the three series, the one for the derivative, $f'(x)$, is the least likely to converge at the endpoints. In fact, it can be shown that if the series for $f'(x)$ converges at the endpoints

$$x = c \pm R$$

then the series for $f(x)$ will also converge there.

9.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding the Center of a Power Series In Exercises 1–4, state where the power series is centered.

1. $\displaystyle\sum_{n=0}^{\infty} nx^n$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 1\cdot 3\cdots(2n-1)}{2^n n!}x^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x-\pi)^{2n}}{(2n)!}$

Finding the Radius of Convergence In Exercises 5–10, find the radius of convergence of the power series.

5. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

6. $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

9. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

Finding the Interval of Convergence In Exercises 11–34, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

11. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

12. $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

14. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

16. $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

17. $\displaystyle\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{6^n}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-5)^n}{3^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n9^n}$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!}$

32. $\displaystyle\sum_{n=1}^{\infty} \left[\frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)}\right]x^{2n+1}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}{4^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

Finding the Radius of Convergence In Exercises 35 and 36, find the radius of convergence of the power series, where $c > 0$ and k is a positive integer.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$

Finding the Interval of Convergence In Exercises 37–40, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

37. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{k}\right)^n, \quad k > 0$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{k(k+1)(k+2)\cdots(k+n-1)x^n}{n!}, \quad k \ge 1$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

Writing an Equivalent Series In Exercises 41–44, write an equivalent series with the index of summation beginning at $n = 1$.

41. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

42. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

Finding Intervals of Convergence In Exercises 45–48, find the intervals of convergence of (a) $f(x)$, (b) $f'(x)$, (c) $f''(x)$, and (d) $\int f(x)\,dx$. Include a check for convergence at the endpoints of the interval.

45. $\displaystyle f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$

46. $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$

47. $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

48. $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}$

WRITING ABOUT CONCEPTS

49. Power Series Define a power series centered at c.

50. Radius of Convergence Describe the radius of convergence of a power series.

51. Interval of Convergence Describe the interval of convergence of a power series.

52. Domain of a Power Series Describe the three basic forms of the domain of a power series.

53. Using a Power Series Describe how to differentiate and integrate a power series with a radius of convergence R. Will the series resulting from the operations of differentiation and integration have a different radius of convergence? Explain.

54. Conditional or Absolute Convergence Give examples that show that the convergence of a power series at an endpoint of its interval of convergence may be either conditional or absolute. Explain your reasoning.

55. Writing a Power Series Write a power series that has the indicated interval of convergence. Explain your reasoning.

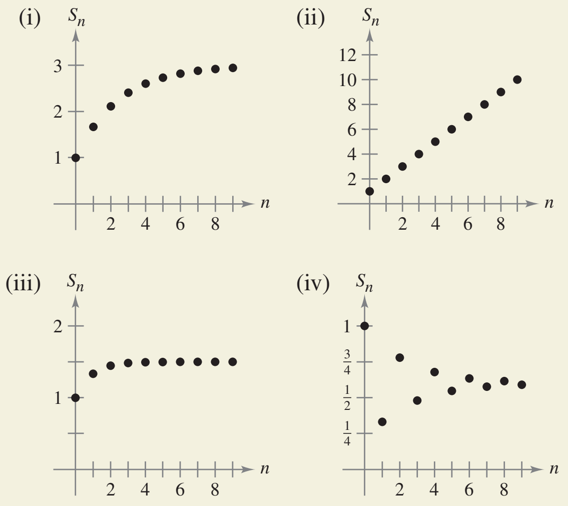
(a) $(-2, 2)$ (b) $(-1, 1]$

(c) $(-1, 0)$ (d) $[-2, 6)$

56. HOW DO YOU SEE IT? Match the graph of the first 10 terms of the sequence of partial sums of the series

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

with the indicated value of the function. [The graphs are labeled (i), (ii), (iii), and (iv).] Explain how you made your choice.



(a) $g(1)$ (b) $g(2)$

(c) $g(3)$ (d) $g(-2)$

57. Using Power Series Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

(a) Find the intervals of convergence of f and g.

(b) Show that $f'(x) = g(x)$.

(c) Show that $g'(x) = -f(x)$.

(d) Identify the functions f and g.

58. Using a Power Series Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Find the interval of convergence of f.

(b) Show that $f'(x) = f(x)$.

(c) Show that $f(0) = 1$.

(d) Identify the function f.

Differential Equation **In Exercises 59–64, show that the function represented by the power series is a solution of the differential equation.**

59. $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $y'' + y = 0$

60. $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $y'' + y = 0$

61. $y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$, $y'' - y = 0$

62. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $y'' - y = 0$

63. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$, $y'' - xy' - y = 0$

64. $y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$, $y'' + x^2 y = 0$

65. Bessel Function The Bessel function of order 0 is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}.$$

(a) Show that the series converges for all x.

(b) Show that the series is a solution of the differential equation $x^2 J_0'' + x J_0' + x^2 J_0 = 0$.

(c) Use a graphing utility to graph the polynomial composed of the first four terms of J_0.

(d) Approximate $\int_0^1 J_0\,dx$ accurate to two decimal places.

66. Bessel Function The Bessel function of order 1 is

$$J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k!(k+1)!}.$$

(a) Show that the series converges for all x.

(b) Show that the series is a solution of the differential equation $x^2 J_1'' + x J_1' + (x^2 - 1) J_1 = 0$.

(c) Use a graphing utility to graph the polynomial composed of the first four terms of J_1.

(d) Show that $J_0'(x) = -J_1(x)$.

67. Investigation The interval of convergence of the geometric series $\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$ is $(-4, 4)$.

(a) Find the sum of the series when $x = \frac{5}{2}$. Use a graphing utility to graph the first six terms of the sequence of partial sums and the horizontal line representing the sum of the series.

(b) Repeat part (a) for $x = -\frac{5}{2}$.

(c) Write a short paragraph comparing the rates of convergence of the partial sums with the sums of the series in parts (a) and (b). How do the plots of the partial sums differ as they converge toward the sum of the series?

(d) Given any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=0}^{N}\left(\frac{5}{4}\right)^n > M.$$

Use a graphing utility to complete the table.

M	10	100	1000	10,000
N				

68. Investigation The interval of convergence of the series $\sum_{n=0}^{\infty}(3x)^n$ is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(a) Find the sum of the series when $x = \frac{1}{6}$. Use a graphing utility to graph the first six terms of the sequence of partial sums and the horizontal line representing the sum of the series.

(b) Repeat part (a) for $x = -\frac{1}{6}$.

(c) Write a short paragraph comparing the rates of convergence of the partial sums with the sums of the series in parts (a) and (b). How do the plots of the partial sums differ as they converge toward the sum of the series?

(d) Given any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=0}^{N}\left(3 \cdot \frac{2}{3}\right)^n > M.$$

Use a graphing utility to complete the table.

M	10	100	1000	10,000
N				

Identifying a Function **In Exercises 69–72, the series represents a well-known function. Use a computer algebra system to graph the partial sum S_{10} and identify the function from the graph.**

69. $f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$

70. $f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

71. $f(x) = \sum_{n=0}^{\infty}(-1)^n x^n, \quad -1 < x < 1$

72. $f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 \le x \le 1$

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. If the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = 2$, then it also converges for $x = -2$.

74. It is possible to find a power series whose interval of convergence is $[0, \infty)$.

75. If the interval of convergence for $\sum_{n=0}^{\infty} a_n x^n$ is $(-1, 1)$, then the interval of convergence for $\sum_{n=0}^{\infty} a_n(x-1)^n$ is $(0, 2)$.

76. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 2$, then

$$\int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}.$$

77. Proof Prove that the power series

$$\sum_{n=0}^{\infty} \frac{(n+p)!}{n!(n+q)!}x^n$$

has a radius of convergence of $R = \infty$ when p and q are positive integers.

78. Using a Power Series Let

$$g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

where the coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for $n \ge 0$.

(a) Find the interval of convergence of the series.

(b) Find an explicit formula for $g(x)$.

79. Using a Power Series Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+3} = c_n$ for $n \ge 0$.

(a) Find the interval of convergence of the series.

(b) Find an explicit formula for $f(x)$.

80. Proof Prove that if the power series $\sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence of R, then $\sum_{n=0}^{\infty} c_n x^{2n}$ has a radius of convergence of $\sqrt{R}$.

81. Proof For $n > 0$, let $R > 0$ and $c_n > 0$. Prove that if the interval of convergence of the series

$$\sum_{n=0}^{\infty} c_n(x - x_0)^n$$

is $[x_0 - R, x_0 + R]$, then the series converges conditionally at $x_0 - R$.

9.9 Representation of Functions by Power Series

- **Find a geometric power series that represents a function.**
- **Construct a power series using series operations.**

JOSEPH FOURIER (1768–1830)

Some of the early work in representing functions by power series was done by the French mathematician Joseph Fourier. Fourier's work is important in the history of calculus, partly because it forced eighteenth-century mathematicians to question the then-prevailing narrow concept of a function. Both Cauchy and Dirichlet were motivated by Fourier's work with series, and in 1837 Dirichlet published the general definition of a function that is used today.

Geometric Power Series

In this section and the next, you will study several techniques for finding a power series that represents a function. Consider the function

$$f(x) = \frac{1}{1-x}.$$

The form of f closely resembles the sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

In other words, when $a = 1$ and $r = x$, a power series representation for $1/(1-x)$, centered at 0, is

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} ar^n \\ &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.\end{aligned}$$

Of course, this series represents $f(x) = 1/(1-x)$ only on the interval $(-1, 1)$, whereas f is defined for all $x \neq 1$, as shown in Figure 9.22. To represent f in another interval, you must develop a different series. For instance, to obtain the power series centered at -1, you could write

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1/2}{1-[(x+1)/2]} = \frac{a}{1-r}$$

which implies that $a = \frac{1}{2}$ and $r = (x+1)/2$. So, for $|x+1| < 2$, you have

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n \\ &= \frac{1}{2}\left[1 + \frac{(x+1)}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^3}{8} + \cdots\right], \quad |x+1| < 2\end{aligned}$$

which converges on the interval $(-3, 1)$.

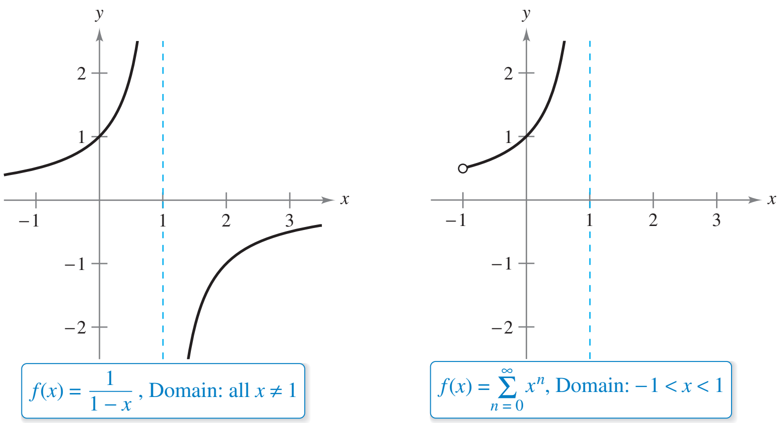


Figure 9.22

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EXAMPLE 1 Finding a Geometric Power Series Centered at 0

Find a power series for $f(x) = \dfrac{4}{x + 2}$, centered at 0.

Solution Writing $f(x)$ in the form $a/(1 - r)$ produces

$$\frac{4}{2 + x} = \frac{2}{1 - (-x/2)} = \frac{a}{1 - r}$$

which implies that $a = 2$ and

$$r = -\frac{x}{2}.$$

So, the power series for $f(x)$ is

$$\begin{aligned}\frac{4}{x + 2} &= \sum_{n=0}^{\infty} ar^n \\ &= \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n \\ &= 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right).\end{aligned}$$

This power series converges when

$$\left|-\frac{x}{2}\right| < 1$$

which implies that the interval of convergence is $(-2, 2)$.

Long Division

$$\begin{array}{r l}
 & 2 - x + \tfrac{1}{2}x^2 - \tfrac{1}{4}x^3 + \cdots \\ \hline
2 + x \,) & 4 \\
 & \underline{4 + 2x} \\
 & -2x \\
 & \underline{-2x - x^2} \\
 & x^2 \\
 & \underline{x^2 + \tfrac{1}{2}x^3} \\
 & -\tfrac{1}{2}x^3 \\
 & \underline{-\tfrac{1}{2}x^3 - \tfrac{1}{4}x^4}
\end{array}$$

Another way to determine a power series for a rational function such as the one in Example 1 is to use long division. For instance, by dividing $2 + x$ into 4, you obtain the result shown at the left.

EXAMPLE 2 Finding a Geometric Power Series Centered at 1

Find a power series for $f(x) = \dfrac{1}{x}$, centered at 1.

Solution Writing $f(x)$ in the form $a/(1 - r)$ produces

$$\frac{1}{x} = \frac{1}{1 - (-x + 1)} = \frac{a}{1 - r}$$

which implies that $a = 1$ and $r = 1 - x = -(x - 1)$. So, the power series for $f(x)$ is

$$\begin{aligned}\frac{1}{x} &= \sum_{n=0}^{\infty} ar^n \\ &= \sum_{n=0}^{\infty} [-(x - 1)]^n \\ &= \sum_{n=0}^{\infty} (-1)^n(x - 1)^n \\ &= 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots.\end{aligned}$$

This power series converges when

$$|x - 1| < 1$$

which implies that the interval of convergence is $(0, 2)$.

Operations with Power Series

The versatility of geometric power series will be shown later in this section, following a discussion of power series operations. These operations, used with differentiation and integration, provide a means of developing power series for a variety of elementary functions. (For simplicity, the operations are stated for a series centered at 0.)

Operations with Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

The operations described above can change the interval of convergence for the resulting series. For example, in the addition shown below, the interval of convergence for the sum is the *intersection* of the intervals of convergence of the two original series.

$$\underbrace{\sum_{n=0}^{\infty} x^n}_{(-1,\,1)} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n}_{\cap\ (-2,\,2)} = \underbrace{\sum_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right) x^n}_{=\ (-1,\,1)}$$

EXAMPLE 3 **Adding Two Power Series**

Find a power series for

$$f(x) = \frac{3x - 1}{x^2 - 1}$$

centered at 0.

Solution Using partial fractions, you can write $f(x)$ as

$$\frac{3x - 1}{x^2 - 1} = \frac{2}{x + 1} + \frac{1}{x - 1}.$$

By adding the two geometric power series

$$\frac{2}{x + 1} = \frac{2}{1 - (-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n, \quad |x| < 1$$

and

$$\frac{1}{x - 1} = \frac{-1}{1 - x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

you obtain the power series shown below.

$$\frac{3x - 1}{x^2 - 1} = \sum_{n=0}^{\infty} [2(-1)^n - 1] x^n$$
$$= 1 - 3x + x^2 - 3x^3 + x^4 - \cdots$$

The interval of convergence for this power series is $(-1, 1)$.

EXAMPLE 4 Finding a Power Series by Integration

Find a power series for

$$f(x) = \ln x$$

centered at 1.

Solution From Example 2, you know that

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n. \qquad \text{Interval of convergence: } (0, 2)$$

Integrating this series produces

$$\begin{aligned}\ln x &= \int \frac{1}{x}\,dx + C \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}.\end{aligned}$$

By letting $x = 1$, you can conclude that $C = 0$. Therefore,

$$\begin{aligned}\ln x &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \\ &= \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots. \qquad \text{Interval of convergence: } (0, 2]\end{aligned}$$

Note that the series converges at $x = 2$. This is consistent with the observation in the preceding section that integration of a power series may alter the convergence at the endpoints of the interval of convergence.

FOR FURTHER INFORMATION To read about finding a power series using integration by parts, see the article "Integration by Parts and Infinite Series" by Shelby J. Kilmer in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

In Section 9.7, Example 4, the fourth-degree Taylor polynomial for the natural logarithmic function

$$\ln x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

was used to approximate $\ln(1.1)$.

$$\begin{aligned}\ln(1.1) &\approx (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 \\ &\approx 0.0953083\end{aligned}$$

You now know from Example 4 in this section that this polynomial represents the first four terms of the power series for $\ln x$. Moreover, using the Alternating Series Remainder, you can determine that the error in this approximation is less than

$$\begin{aligned}|R_4| &\le |a_5| \\ &= \frac{1}{5}(0.1)^5 \\ &= 0.000002.\end{aligned}$$

During the seventeenth and eighteenth centuries, mathematical tables for logarithms and values of other transcendental functions were computed in this manner. Such numerical techniques are far from outdated, because it is precisely by such means that many modern calculating devices are programmed to evaluate transcendental functions.

EXAMPLE 5 Finding a Power Series by Integration

See LarsonCalculus.com for an interactive version of this type of example.

Find a power series for

$$g(x) = \arctan x$$

centered at 0.

Solution Because $D_x[\arctan x] = 1/(1 + x^2)$, you can use the series

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n. \qquad \text{Interval of convergence: } (-1, 1)$$

Substituting x^2 for x produces

$$f(x^2) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Finally, by integrating, you obtain

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2}\,dx + C \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} && \text{Let } x = 0, \text{ then } C = 0. \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots. && \text{Interval of convergence: } (-1, 1)
\end{aligned}$$

SRINIVASA RAMANUJAN (1887–1920)

Series that can be used to approximate π have interested mathematicians for the past 300 years. An amazing series for approximating $1/\pi$ was discovered by the Indian mathematician Srinivasa Ramanujan in 1914 (see Exercise 61). Each successive term of Ramanujan's series adds roughly eight more correct digits to the value of $1/\pi$. For more information about Ramanujan's work, see the article "Ramanujan and Pi" by Jonathan M. Borwein and Peter B. Borwein in *Scientific American*.

See LarsonCalculus.com to read more of this biography.

It can be shown that the power series developed for $\arctan x$ in Example 5 also converges (to $\arctan x$) for $x = \pm 1$. For instance, when $x = 1$, you can write

$$\begin{aligned}
\arctan 1 &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\
&= \frac{\pi}{4}.
\end{aligned}$$

However, this series (developed by James Gregory in 1671) does not give us a practical way of approximating π because it converges so slowly that hundreds of terms would have to be used to obtain reasonable accuracy. Example 6 shows how to use *two* different arctangent series to obtain a very good approximation of π using only a few terms. This approximation was developed by John Machin in 1706.

FOR FURTHER INFORMATION To read about other methods for approximating π, see the article "Two Methods for Approximating π" by Chien-Lih Hwang in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

EXAMPLE 6 Approximating π with a Series

Use the trigonometric identity

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

to approximate the number π [see Exercise 46(b)].

Solution By using only five terms from each of the series for $\arctan(1/5)$ and $\arctan(1/239)$, you obtain

$$4\left(4 \arctan \frac{1}{5} - \arctan \frac{1}{239}\right) \approx 3.1415926$$

which agrees with the exact value of π with an error of less than 0.0000001.

The Granger Collection

9.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Geometric Power Series In Exercises 1–4, find a geometric power series for the function, centered at 0, (a) by the technique shown in Examples 1 and 2 and (b) by long division.

1. $f(x) = \dfrac{1}{4-x}$

2. $f(x) = \dfrac{1}{2+x}$

3. $f(x) = \dfrac{4}{3+x}$

4. $f(x) = \dfrac{2}{5-x}$

Finding a Power Series In Exercises 5–16, find a power series for the function, centered at c, and determine the interval of convergence.

5. $f(x) = \dfrac{1}{3-x}, \quad c = 1$

6. $f(x) = \dfrac{2}{6-x}, \quad c = -2$

7. $f(x) = \dfrac{1}{1-3x}, \quad c = 0$

8. $h(x) = \dfrac{1}{1-5x}, \quad c = 0$

9. $g(x) = \dfrac{5}{2x-3}, \quad c = -3$

10. $f(x) = \dfrac{3}{2x-1}, \quad c = 2$

11. $f(x) = \dfrac{3}{3x+4}, \quad c = 0$

12. $f(x) = \dfrac{4}{3x+2}, \quad c = 3$

13. $g(x) = \dfrac{4x}{x^2+2x-3}, \quad c = 0$

14. $g(x) = \dfrac{3x-8}{3x^2+5x-2}, \quad c = 0$

15. $f(x) = \dfrac{2}{1-x^2}, \quad c = 0$

16. $f(x) = \dfrac{5}{5+x^2}, \quad c = 0$

Using a Power Series In Exercises 17–26, use the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

17. $h(x) = \dfrac{-2}{x^2-1} = \dfrac{1}{1+x} + \dfrac{1}{1-x}$

18. $h(x) = \dfrac{x}{x^2-1} = \dfrac{1}{2(1+x)} - \dfrac{1}{2(1-x)}$

19. $f(x) = -\dfrac{1}{(x+1)^2} = \dfrac{d}{dx}\left[\dfrac{1}{x+1}\right]$

20. $f(x) = \dfrac{2}{(x+1)^3} = \dfrac{d^2}{dx^2}\left[\dfrac{1}{x+1}\right]$

21. $f(x) = \ln(x+1) = \displaystyle\int \frac{1}{x+1}\,dx$

22. $f(x) = \ln(1-x^2) = \displaystyle\int \frac{1}{1+x}\,dx - \int \frac{1}{1-x}\,dx$

23. $g(x) = \dfrac{1}{x^2+1}$

24. $f(x) = \ln(x^2+1)$

25. $h(x) = \dfrac{1}{4x^2+1}$

26. $f(x) = \arctan 2x$

Graphical and Numerical Analysis In Exercises 27 and 28, let

$$S_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n}.$$

Use a graphing utility to confirm the inequality graphically. Then complete the table to confirm the inequality numerically.

x	0.0	0.2	0.4	0.6	0.8	1.0
S_n						
$\ln(x+1)$						
S_{n+1}						

27. $S_2 \le \ln(x+1) \le S_3$

28. $S_4 \le \ln(x+1) \le S_5$

Approximating a Sum In Exercises 29 and 30, (a) graph several partial sums of the series, (b) find the sum of the series and its radius of convergence, (c) use 50 terms of the series to approximate the sum when $x = 0.5$, and (d) determine what the approximation represents and how good the approximation is.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Approximating a Value In Exercises 31–34, use the series for $f(x) = \arctan x$ to approximate the value, using $R_N \le 0.001$.

31. $\arctan \dfrac{1}{4}$

32. $\displaystyle\int_0^{3/4} \arctan x^2\,dx$

33. $\displaystyle\int_0^{1/2} \frac{\arctan x^2}{x}\,dx$

34. $\displaystyle\int_0^{1/2} x^2 \arctan x\,dx$

Using a Power Series In Exercises 35–38, use the power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Find the series representation of the function and determine its interval of convergence.

35. $f(x) = \dfrac{1}{(1-x)^2}$

36. $f(x) = \dfrac{x}{(1-x)^2}$

37. $f(x) = \dfrac{1+x}{(1-x)^2}$

38. $f(x) = \dfrac{x(1+x)}{(1-x)^2}$

39. Probability A fair coin is tossed repeatedly. The probability that the first head occurs on the nth toss is $P(n) = \left(\frac{1}{2}\right)^n$. When this game is repeated many times, the average number of tosses required until the first head occurs is

$$E(n) = \sum_{n=1}^{\infty} nP(n).$$

(This value is called the *expected value of* n.) Use the results of Exercises 35–38 to find $E(n)$. Is the answer what you expected? Why or why not?

40. Finding the Sum of a Series Use the results of Exercises 35–38 to find the sum of each series.

(a) $\frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ (b) $\frac{1}{10}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n$

Writing In Exercises 41–44, explain how to use the geometric series

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

to find the series for the function. Do not find the series.

41. $f(x) = \dfrac{1}{1+x}$ **42.** $f(x) = \dfrac{1}{1-x^2}$

43. $f(x) = \dfrac{5}{1+x}$ **44.** $f(x) = \ln(1-x)$

45. Proof Prove that

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$$

for $xy \neq 1$ provided the value of the left side of the equation is between $-\pi/2$ and $\pi/2$.

46. Verifying an Identity Use the result of Exercise 45 to verify each identity.

(a) $\arctan \dfrac{120}{119} - \arctan \dfrac{1}{239} = \dfrac{\pi}{4}$

(b) $4 \arctan \dfrac{1}{5} - \arctan \dfrac{1}{239} = \dfrac{\pi}{4}$

[*Hint:* Use Exercise 45 twice to find $4 \arctan \frac{1}{5}$. Then use part (a).]

Approximating Pi In Exercises 47 and 48, (a) verify the given equation, and (b) use the equation and the series for the arctangent to approximate π to two-decimal-place accuracy.

47. $2 \arctan \dfrac{1}{2} - \arctan \dfrac{1}{7} = \dfrac{\pi}{4}$

48. $\arctan \dfrac{1}{2} + \arctan \dfrac{1}{3} = \dfrac{\pi}{4}$

Finding the Sum of a Series In Exercises 49–54, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.

49. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{2^n n}$ **50.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{3^n n}$

51. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{2^n}{5^n n}$ **52.** $\displaystyle\sum_{n=0}^{\infty} (-1)^{n}\frac{1}{2n+1}$

53. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n}\frac{1}{2^{2n+1}(2n+1)}$

54. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{3^{2n-1}(2n-1)}$

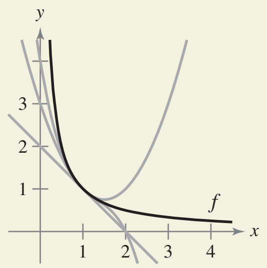
WRITING ABOUT CONCEPTS

55. Using Series One of the series in Exercises 49–54 converges to its sum at a much lower rate than the other five series. Which is it? Explain why this series converges so slowly. Use a graphing utility to illustrate the rate of convergence.

56. Radius of Convergence The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is 3. What is the radius of convergence of the series $\sum_{n=1}^{\infty} na_n x^{n-1}$? Explain.

57. Convergence of a Power Series The power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x+1| < 4$. What can you conclude about the series $\sum_{n=0}^{\infty} a_n \dfrac{x^{n+1}}{n+1}$? Explain.

58. HOW DO YOU SEE IT? The graphs show first-, second-, and third-degree polynomial approximations P_1, P_2, and P_3 of a function f. Label the graphs of P_1, P_2, and P_3. To print an enlarged copy of the graph, go to *MathGraphs.com.*



Finding the Sum of a Series In Exercises 59 and 60, find the sum of the series.

59. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$ **60.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!}$

61. Ramanujan and Pi Use a graphing utility to show that

$$\frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103+26{,}390n)}{(n!)396^{4n}} = \frac{1}{\pi}.$$

62. Find the Error Describe why the statement is incorrect.

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\left(1+\frac{1}{5}\right)x^n$$

9.10 Taylor and Maclaurin Series

- **Find a Taylor or Maclaurin series for a function.**
- **Find a binomial series.**
- **Use a basic list of Taylor series to find other Taylor series.**

Taylor Series and Maclaurin Series

In Section 9.9, you derived power series for several functions using geometric series with term-by-term differentiation or integration. In this section, you will study a *general* procedure for deriving the power series for a function that has derivatives of all orders. The next theorem gives the form that *every* convergent power series must take.

REMARK Be sure you understand Theorem 9.22. The theorem says that *if a power series converges to* $f(x)$, then the series must be a Taylor series. The theorem does *not* say that every series formed with the Taylor coefficients $a_n = f^{(n)}(c)/n!$ will converge to $f(x)$.

THEOREM 9.22 The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \Sigma\, a_n(x-c)^n$ for all x in an open interval I containing c, then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdot\cdot\cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdot\cdot\cdot.$$

Proof Consider a power series $\Sigma\, a_n(x-c)^n$ that has a radius of convergence R. Then, by Theorem 9.21, you know that the nth derivative of f exists for $|x-c| < R$, and by successive differentiation you obtain the following.

$$\begin{aligned}
f^{(0)}(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdot\cdot\cdot \\
f^{(1)}(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdot\cdot\cdot \\
f^{(2)}(x) &= 2a_2 + 3!a_3(x-c) + 4\cdot 3a_4(x-c)^2 + \cdot\cdot\cdot \\
f^{(3)}(x) &= 3!a_3 + 4!a_4(x-c) + \cdot\cdot\cdot \\
&\ \vdots \\
f^{(n)}(x) &= n!a_n + (n+1)!a_{n+1}(x-c) + \cdot\cdot\cdot
\end{aligned}$$

Evaluating each of these derivatives at $x = c$ yields

$$\begin{aligned}
f^{(0)}(c) &= 0!a_0 \\
f^{(1)}(c) &= 1!a_1 \\
f^{(2)}(c) &= 2!a_2 \\
f^{(3)}(c) &= 3!a_3
\end{aligned}$$

and, in general, $f^{(n)}(c) = n!a_n$. By solving for a_n, you find that the coefficients of the power series representation of $f(x)$ are

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

COLIN MACLAURIN (1698–1746)

The development of power series to represent functions is credited to the combined work of many seventeenth- and eighteenth-century mathematicians. Gregory, Newton, John and James Bernoulli, Leibniz, Euler, Lagrange, Wallis, and Fourier all contributed to this work. However, the two names that are most commonly associated with power series are Brook Taylor (1685–1731) and Colin Maclaurin.

See LarsonCalculus.com to read more of this biography.

Notice that the coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at c as defined in Section 9.7. For this reason, the series is called the **Taylor series** for $f(x)$ at c.

Bettmann/Corbis

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at c.** Moreover, if $c = 0$, then the series is the **Maclaurin series for f.**

When you know the pattern for the coefficients of the Taylor polynomials for a function, you can extend the pattern easily to form the corresponding Taylor series. For instance, in Example 4 in Section 9.7, you found the fourth Taylor polynomial for $\ln x$, centered at 1, to be

$$P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

From this pattern, you can obtain the Taylor series for $\ln x$ centered at $c = 1$,

$$(x-1) - \frac{1}{2}(x-1)^2 + \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n + \cdots.$$

EXAMPLE 1 Forming a Power Series

Use the function

$$f(x) = \sin x$$

to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

and determine the interval of convergence.

Solution Successive differentiation of $f(x)$ yields

$$\begin{aligned}
f(x) &= \sin x & f(0) &= \sin 0 = 0\\
f'(x) &= \cos x & f'(0) &= \cos 0 = 1\\
f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0\\
f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -\cos 0 = -1\\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0\\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= \cos 0 = 1
\end{aligned}$$

and so on. The pattern repeats after the third derivative. So, the power series is as follows.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots\\
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &= 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6\\
&\quad + \frac{(-1)}{7!}x^7 + \cdots\\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{aligned}$$

By the Ratio Test, you can conclude that this series converges for all x.

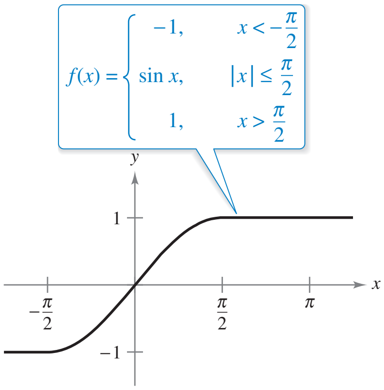


Figure 9.23

Notice that in Example 1, you cannot conclude that the power series converges to $\sin x$ for all x. You can simply conclude that the power series converges to some function, but you are not sure what function it is. This is a subtle, but important, point in dealing with Taylor or Maclaurin series. To persuade yourself that the series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

might converge to a function other than f, remember that the derivatives are being evaluated at a single point. It can easily happen that another function will agree with the values of $f^{(n)}(x)$ when $x = c$ and disagree at other x-values. For instance, the power series (centered at 0) for the function f shown in Figure 9.23 is the same series as in Example 1. You know that the series converges for all x, and yet it obviously cannot converge to both $f(x)$ and $\sin x$ for all x.

Let f have derivatives of all orders in an open interval I centered at c. The Taylor series for f may fail to converge for some x in I. Or, even when it is convergent, it may fail to have $f(x)$ as its sum. Nevertheless, Theorem 9.19 tells us that for each n,

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

Note that in this remainder formula, the particular value of z that makes the remainder formula true depends on the values of x and n. If $R_n \to 0$, then the next theorem tells us that the Taylor series for f actually converges to $f(x)$ for all x in I.

THEOREM 9.23 Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

Proof For a Taylor series, the nth partial sum coincides with the nth Taylor polynomial. That is, $S_n(x) = P_n(x)$. Moreover, because

$$P_n(x) = f(x) - R_n(x)$$

it follows that

$$\begin{aligned} \lim_{n\to\infty} S_n(x) &= \lim_{n\to\infty} P_n(x) \\ &= \lim_{n\to\infty} [f(x) - R_n(x)] \\ &= f(x) - \lim_{n\to\infty} R_n(x). \end{aligned}$$

So, for a given x, the Taylor series (the sequence of partial sums) converges to $f(x)$ if and only if $R_n(x) \to 0$ as $n \to \infty$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Stated another way, Theorem 9.23 says that a power series formed with Taylor coefficients $a_n = f^{(n)}(c)/n!$ converges to the function from which it was derived at precisely those values for which the remainder approaches 0 as $n \to \infty$.

In Example 1, you derived the power series from the sine function and you also concluded that the series converges to some function on the entire real number line. In Example 2, you will see that the series actually converges to $\sin x$. The key observation is that although the value of z is not known, it is possible to obtain an upper bound for

$$\left|f^{(n+1)}(z)\right|.$$

EXAMPLE 2 A Convergent Maclaurin Series

Show that the Maclaurin series for

$$f(x) = \sin x$$

converges to $\sin x$ for all x.

Solution Using the result in Example 1, you need to show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdot \cdot \cdot + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdot \cdot \cdot$$

is true for all x. Because

$$f^{(n+1)}(x) = \pm\sin x$$

or

$$f^{(n+1)}(x) = \pm\cos x$$

you know that $\left|f^{(n+1)}(z)\right| \le 1$ for every real number z. Therefore, for any fixed x, you can apply Taylor's Theorem (Theorem 9.19) to conclude that

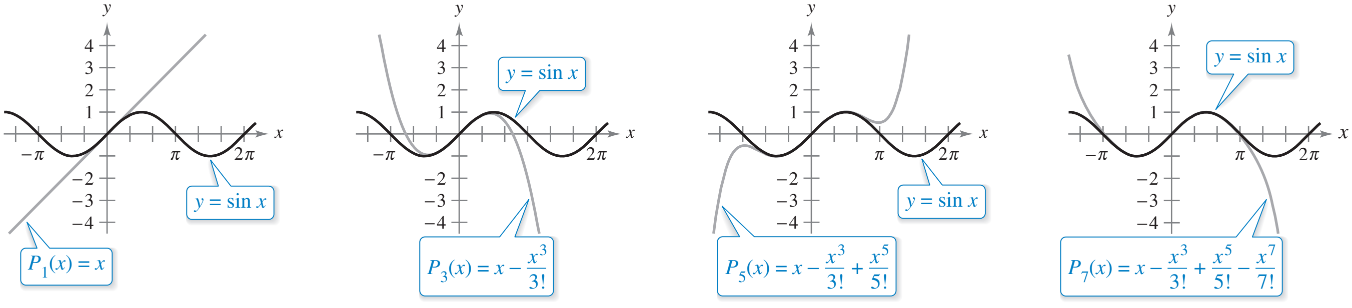
$$0 \le \left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

From the discussion in Section 9.1 regarding the relative rates of convergence of exponential and factorial sequences, it follows that for a fixed x

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Finally, by the Squeeze Theorem, it follows that for all x, $R_n(x) \to 0$ as $n \to \infty$. So, by Theorem 9.23, the Maclaurin series for $\sin x$ converges to $\sin x$ for all x.

Figure 9.24 visually illustrates the convergence of the Maclaurin series for $\sin x$ by comparing the graphs of the Maclaurin polynomials $P_1(x)$, $P_3(x)$, $P_5(x)$, and $P_7(x)$ with the graph of the sine function. Notice that as the degree of the polynomial increases, its graph more closely resembles that of the sine function.



As n increases, the graph of P_n more closely resembles the sine function.
Figure 9.24

The guidelines for finding a Taylor series for $f(x)$ at c are summarized below.

GUIDELINES FOR FINDING A TAYLOR SERIES

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c), f'(c), f''(c), f'''(c), \cdot \cdot \cdot, f^{(n)}(c), \cdot \cdot \cdot$$

   Try to recognize a pattern in these numbers.
2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdot \cdot \cdot.$$

3. Within this interval of convergence, determine whether the series converges to $f(x)$.

REMARK When you have difficulty recognizing a pattern, remember that you can use Theorem 9.22 to find the Taylor series. Also, you can try using the coefficients of a known Taylor or Maclaurin series, as shown in Example 3.

The direct determination of Taylor or Maclaurin coefficients using successive differentiation can be difficult, and the next example illustrates a shortcut for finding the coefficients indirectly—using the coefficients of a known Taylor or Maclaurin series.

EXAMPLE 3 Maclaurin Series for a Composite Function

Find the Maclaurin series for

$$f(x) = \sin x^2.$$

Solution To find the coefficients for this Maclaurin series directly, you must calculate successive derivatives of $f(x) = \sin x^2$. By calculating just the first two,

$$f'(x) = 2x \cos x^2$$

and

$$f''(x) = -4x^2 \sin x^2 + 2 \cos x^2$$

you can see that this task would be quite cumbersome. Fortunately, there is an alternative. First, consider the Maclaurin series for $\sin x$ found in Example 1.

$$\begin{aligned} g(x) &= \sin x \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdot \cdot \cdot \end{aligned}$$

Now, because $\sin x^2 = g(x^2)$, you can substitute x^2 for x in the series for $\sin x$ to obtain

$$\begin{aligned} \sin x^2 &= g(x^2) \\ &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdot \cdot \cdot. \end{aligned}$$

Be sure to understand the point illustrated in Example 3. Because direct computation of Taylor or Maclaurin coefficients can be tedious, the most practical way to find a Taylor or Maclaurin series is to develop power series for a *basic list* of elementary functions. From this list, you can determine power series for other functions by the operations of addition, subtraction, multiplication, division, differentiation, integration, and composition with known power series.

Binomial Series

Before presenting the basic list for elementary functions, you will develop one more series—for a function of the form $f(x) = (1 + x)^k$. This produces the **binomial series.**

EXAMPLE 4 Binomial Series

Find the Maclaurin series for $f(x) = (1 + x)^k$ and determine its radius of convergence. Assume that k is not a positive integer and $k \neq 0$.

Solution By successive differentiation, you have

$$\begin{aligned} f(x) &= (1+x)^k & f(0) &= 1 \\ f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\ f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\ f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\ &\vdots & &\vdots \\ f^{(n)}(x) &= k \cdots (k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\ldots(k-n+1) \end{aligned}$$

which produces the series

$$1 + kx + \frac{k(k-1)x^2}{2} + \cdots + \frac{k(k-1)\cdots(k-n+1)x^n}{n!} + \cdots.$$

Because $a_{n+1}/a_n \to 1$, you can apply the Ratio Test to conclude that the radius of convergence is $R = 1$. So, the series converges to some function in the interval $(-1, 1)$.

Note that Example 4 shows that the Taylor series for $(1 + x)^k$ converges to some function in the interval $(-1, 1)$. However, the example does not show that the series actually converges to $(1 + x)^k$. To do this, you could show that the remainder $R_n(x)$ converges to 0, as illustrated in Example 2. You now have enough information to find a binomial series for a function, as shown in the next example.

EXAMPLE 5 Finding a Binomial Series

Find the power series for $f(x) = \sqrt[3]{1 + x}$.

Solution Using the binomial series

$$(1 + x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

let $k = \frac{1}{3}$ and write

$$(1 + x)^{1/3} = 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdots$$

which converges for $-1 \le x \le 1$.

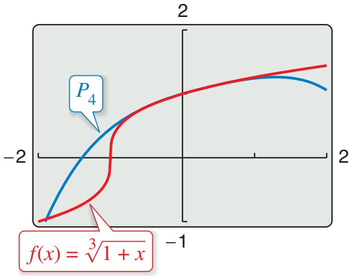


Figure 9.25

TECHNOLOGY Use a graphing utility to confirm the result in Example 5. When you graph the functions

$$f(x) = (1 + x)^{1/3}$$

and

$$P_4(x) = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$$

in the same viewing window, you should obtain the result shown in Figure 9.25.

Deriving Taylor Series from a Basic List

The list below provides the power series for several elementary functions with the corresponding intervals of convergence.

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$*

* The convergence at $x = \pm 1$ depends on the value of k.

Note that the binomial series is valid for noninteger values of k. Also, when k is a positive integer, the binomial series reduces to a simple binomial expansion.

EXAMPLE 6 Deriving a Power Series from a Basic List

Find the power series for

$$f(x) = \cos \sqrt{x}.$$

Solution Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

you can replace x by

$$\sqrt{x}$$

to obtain the series

$$\cos \sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges for all x in the domain of $\cos \sqrt{x}$—that is, for $x \ge 0$. ■

Power series can be multiplied and divided like polynomials. After finding the first few terms of the product (or quotient), you may be able to recognize a pattern.

EXAMPLE 7 Multiplication of Power Series

Find the first three nonzero terms in the Maclaurin series $e^x \arctan x$.

Solution Using the Maclaurin series for e^x and $\arctan x$ in the table, you have

$$e^x \arctan x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right).$$

Multiply these expressions and collect like terms as you would in multiplying polynomials.

$$\begin{array}{rrrrrl}
1 + x + & \frac{1}{2}x^2 + & \frac{1}{6}x^3 + & \frac{1}{24}x^4 + \cdots & & \\
x & & -\frac{1}{3}x^3 & & +\frac{1}{5}x^5 - \cdots & \\
\hline
x + & x^2 + & \frac{1}{2}x^3 + & \frac{1}{6}x^4 + & \frac{1}{24}x^5 + \cdots & \\
 & & -\frac{1}{3}x^3 - & \frac{1}{3}x^4 - & \frac{1}{6}x^5 - \cdots & \\
 & & & & +\frac{1}{5}x^5 + \cdots & \\
\hline
x + & x^2 + & \frac{1}{6}x^3 - & \frac{1}{6}x^4 + & \frac{3}{40}x^5 + \cdots &
\end{array}$$

So, $e^x \arctan x = x + x^2 + \frac{1}{6}x^3 + \cdots$.

EXAMPLE 8 Division of Power Series

Find the first three nonzero terms in the Maclaurin series $\tan x$.

Solution Using the Maclaurin series for $\sin x$ and $\cos x$ in the table, you have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}.$$

Divide using long division.

$$\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\hline
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\
 & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\
\cline{2-2}
 & \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\
 & \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\
\cline{2-2}
 & \frac{2}{15}x^5 + \cdots
\end{array}$$

So, $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$.

EXAMPLE 9 A Power Series for $\sin^2 x$

Find the power series for

$$f(x) = \sin^2 x.$$

Solution Consider rewriting $\sin^2 x$ as

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

Now, use the series for $\cos x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\cos 2x = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \frac{2^8}{8!}x^8 - \cdots$$

$$-\frac{1}{2}\cos 2x = -\frac{1}{2} + \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots$$

$$\frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2} + \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots$$

So, the series for $f(x) = \sin^2 x$ is

$$\sin^2 x = \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots.$$

This series converges for $-\infty < x < \infty$.

As mentioned in the preceding section, power series can be used to obtain tables of values of transcendental functions. They are also useful for estimating the values of definite integrals for which antiderivatives cannot be found. The next example demonstrates this use.

EXAMPLE 10 Power Series Approximation of a Definite Integral

See LarsonCalculus.com for an interactive version of this type of example.

Use a power series to approximate

$$\int_0^1 e^{-x^2}\,dx$$

with an error of less than 0.01.

Solution Replacing x with $-x^2$ in the series for e^x produces the following.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

Summing the first four terms, you have

$$\int_0^1 e^{-x^2}\,dx \approx 0.74$$

which, by the Alternating Series Test, has an error of less than $\frac{1}{216} \approx 0.005$.

9.10 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Taylor Series In Exercises 1–12, use the definition of Taylor series to find the Taylor series, centered at c, for the function.

1. $f(x) = e^{2x}, \quad c = 0$
2. $f(x) = e^{-4x}, \quad c = 0$
3. $f(x) = \cos x, \quad c = \dfrac{\pi}{4}$
4. $f(x) = \sin x, \quad c = \dfrac{\pi}{4}$
5. $f(x) = \dfrac{1}{x}, \quad c = 1$
6. $f(x) = \dfrac{1}{1-x}, \quad c = 2$
7. $f(x) = \ln x, \quad c = 1$
8. $f(x) = e^x, \quad c = 1$
9. $f(x) = \sin 3x, \quad c = 0$
10. $f(x) = \ln(x^2+1), \quad c = 0$
11. $f(x) = \sec x, \quad c = 0$ (first three nonzero terms)
12. $f(x) = \tan x, \quad c = 0$ (first three nonzero terms)

Proof In Exercises 13–16, prove that the Maclaurin series for the function converges to the function for all x.

13. $f(x) = \cos x$
14. $f(x) = e^{-2x}$
15. $f(x) = \sinh x$
16. $f(x) = \cosh x$

Using a Binomial Series In Exercises 17–26, use the binomial series to find the Maclaurin series for the function.

17. $f(x) = \dfrac{1}{(1+x)^2}$
18. $f(x) = \dfrac{1}{(1+x)^4}$
19. $f(x) = \dfrac{1}{\sqrt{1-x}}$
20. $f(x) = \dfrac{1}{\sqrt{1-x^2}}$
21. $f(x) = \dfrac{1}{\sqrt{4+x^2}}$
22. $f(x) = \dfrac{1}{(2+x)^3}$
23. $f(x) = \sqrt{1+x}$
24. $f(x) = \sqrt[4]{1+x}$
25. $f(x) = \sqrt{1+x^2}$
26. $f(x) = \sqrt{1+x^3}$

Finding a Maclaurin Series In Exercises 27–40, find the Maclaurin series for the function. Use the table of power series for elementary functions on page 670.

27. $f(x) = e^{x^2/2}$
28. $g(x) = e^{-3x}$
29. $f(x) = \ln(1+x)$
30. $f(x) = \ln(1+x^2)$
31. $g(x) = \sin 3x$
32. $f(x) = \sin \pi x$
33. $f(x) = \cos 4x$
34. $f(x) = \cos \pi x$
35. $f(x) = \cos x^{3/2}$
36. $g(x) = 2 \sin x^3$
37. $f(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$
38. $f(x) = e^x + e^{-x} = 2 \cosh x$
39. $f(x) = \cos^2 x$
40. $f(x) = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\left(\textit{Hint:}\text{ Integrate the series for } \dfrac{1}{\sqrt{x^2+1}}.\right)$

Finding a Maclaurin Series In Exercises 41–44, find the Maclaurin series for the function. (See Examples 7 and 8.)

41. $f(x) = x \sin x$
42. $h(x) = x \cos x$
43. $g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$
44. $f(x) = \begin{cases} \dfrac{\arcsin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Verifying a Formula In Exercises 45 and 46, use a power series and the fact that $i^2 = -1$ to verify the formula.

45. $g(x) = \dfrac{1}{2i}(e^{ix} - e^{-ix}) = \sin x$
46. $g(x) = \dfrac{1}{2}(e^{ix} + e^{-ix}) = \cos x$

Finding Terms of a Maclaurin Series In Exercises 47–52, find the first four nonzero terms of the Maclaurin series for the function by multiplying or dividing the appropriate power series. Use the table of power series for elementary functions on page 670. Use a graphing utility to graph the function and its corresponding polynomial approximation.

47. $f(x) = e^x \sin x$
48. $g(x) = e^x \cos x$
49. $h(x) = \cos x \ln(1+x)$
50. $f(x) = e^x \ln(1+x)$
51. $g(x) = \dfrac{\sin x}{1+x}$
52. $f(x) = \dfrac{e^x}{1+x}$

Finding a Maclaurin Series In Exercises 53 and 54, find a Maclaurin series for $f(x)$.

53. $f(x) = \displaystyle\int_0^x (e^{-t^2} - 1)\, dt$
54. $f(x) = \displaystyle\int_0^x \sqrt{1+t^3}\, dt$

Verifying a Sum In Exercises 55–58, verify the sum. Then use a graphing utility to approximate the sum with an error of less than 0.0001.

55. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$
56. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n+1)!}\right] = \sin 1$
57. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$
58. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n!}\right) = \frac{e-1}{e}$

Finding a Limit In Exercises 59–62, use the series representation of the function f to find $\lim\limits_{x \to 0} f(x)$ (if it exists).

59. $f(x) = \dfrac{1 - \cos x}{x}$
60. $f(x) = \dfrac{\sin x}{x}$
61. $f(x) = \dfrac{e^x - 1}{x}$
62. $f(x) = \dfrac{\ln(x+1)}{x}$

Approximating an Integral **In Exercises 63–70, use a power series to approximate the value of the integral with an error of less than 0.0001. (In Exercises 65 and 67, assume that the integrand is defined as 1 when $x = 0$.)**

63. $\displaystyle\int_0^1 e^{-x^3}\,dx$

64. $\displaystyle\int_0^{1/4} x\ln(x+1)\,dx$

65. $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$

66. $\displaystyle\int_0^1 \cos x^2\,dx$

67. $\displaystyle\int_0^{1/2} \frac{\arctan x}{x}\,dx$

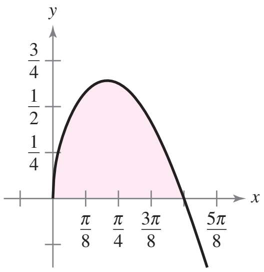
68. $\displaystyle\int_0^{1/2} \arctan x^2\,dx$

69. $\displaystyle\int_{0.1}^{0.3} \sqrt{1+x^3}\,dx$

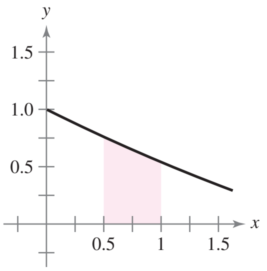
70. $\displaystyle\int_0^{0.2} \sqrt{1+x^2}\,dx$

Area **In Exercises 71 and 72, use a power series to approximate the area of the region. Use a graphing utility to verify the result.**

71. $\displaystyle\int_0^{\pi/2} \sqrt{x}\cos x\,dx$

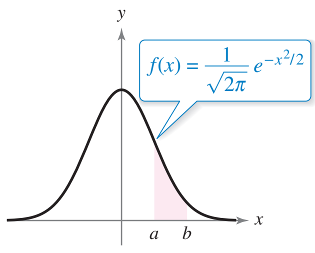


72. $\displaystyle\int_{0.5}^{1} \cos\sqrt{x}\,dx$



Probability **In Exercises 73 and 74, approximate the normal probability with an error of less than 0.0001, where the probability is given by**

$$P(a < x < b) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx.$$



73. $P(0 < x < 1)$

74. $P(1 < x < 2)$

Finding a Taylor Polynomial Using Technology **In Exercises 75–78, use a computer algebra system to find the fifth-degree Taylor polynomial, centered at c, for the function. Graph the function and the polynomial. Use the graph to determine the largest interval on which the polynomial is a reasonable approximation of the function.**

75. $f(x) = x\cos 2x, \quad c = 0$

76. $f(x) = \sin\dfrac{x}{2}\ln(1+x), \quad c = 0$

77. $g(x) = \sqrt{x}\ln x, \quad c = 1$

78. $h(x) = \sqrt[3]{x}\arctan x, \quad c = 1$

WRITING ABOUT CONCEPTS

79. Taylor Series State the guidelines for finding a Taylor series.

80. Binomial Series Define the binomial series. What is its radius of convergence?

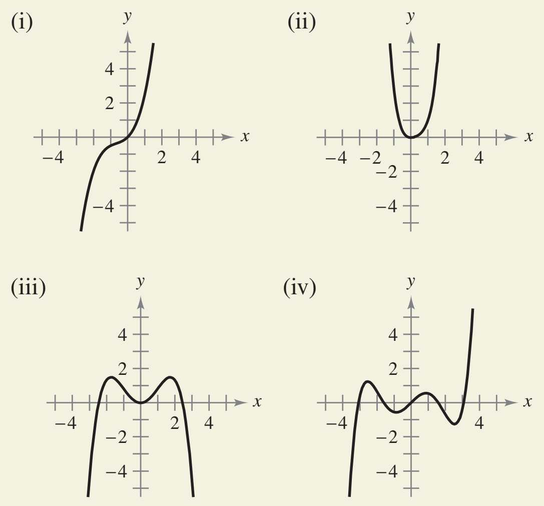
81. Finding a Series Explain how to use the series

$$g(x) = e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

to find the series for each function. Do not find the series.

(a) $f(x) = e^{-x}$ (b) $f(x) = e^{3x}$ (c) $f(x) = xe^x$

82. HOW DO YOU SEE IT? Match the polynomial with its graph. [The graphs are labeled (i), (ii), (iii), and (iv).] Factor a common factor from each polynomial and identify the function approximated by the remaining Taylor polynomial.



(a) $y = x^2 - \dfrac{x^4}{3!}$ (b) $y = x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!}$

(c) $y = x + x^2 + \dfrac{x^3}{2!}$ (d) $y = x^2 - x^3 + x^4$

83. Projectile Motion A projectile fired from the ground follows the trajectory given by

$$y = \left(\tan\theta - \frac{g}{kv_0\cos\theta}\right)x - \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_0\cos\theta}\right)$$

where v_0 is the initial speed, θ is the angle of projection, g is the acceleration due to gravity, and k is the drag factor caused by air resistance. Using the power series representation

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1$$

verify that the trajectory can be rewritten as

$$y = (\tan\theta)x + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{kgx^3}{3v_0^3\cos^3\theta} + \frac{k^2gx^4}{4v_0^4\cos^4\theta} + \cdots.$$

84. Projectile Motion

Use the result of Exercise 83 to determine the series for the path of a projectile launched from ground level at an angle of $\theta = 60°$, with an initial speed of $v_0 = 64$ feet per second and a drag factor of $k = \frac{1}{16}$.

85. Investigation Consider the function f defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(a) Sketch a graph of the function.

(b) Use the alternative form of the definition of the derivative (Section 2.1) and L'Hôpital's Rule to show that $f'(0) = 0$. [By continuing this process, it can be shown that $f^{(n)}(0) = 0$ for $n > 1$.]

(c) Using the result in part (b), find the Maclaurin series for f. Does the series converge to f?

86. Investigation

(a) Find the power series centered at 0 for the function

$$f(x) = \frac{\ln(x^2+1)}{x^2}.$$

(b) Use a graphing utility to graph f and the eighth-degree Taylor polynomial $P_8(x)$ for f.

(c) Complete the table, where

$$F(x) = \int_0^x \frac{\ln(t^2+1)}{t^2}\,dt \quad \text{and} \quad G(x) = \int_0^x P_8(t)\,dt.$$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$						
$G(x)$						

(d) Describe the relationship between the graphs of f and P_8 and the results given in the table in part (c).

87. Proof Prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real x.

88. Finding a Maclaurin Series Find the Maclaurin series for

$$f(x) = \ln\frac{1+x}{1-x}$$

and determine its radius of convergence. Use the first four terms of the series to approximate ln 3.

Evaluating a Binomial Coefficient In Exercises 89–92, evaluate the binomial coefficient using the formula

$$\binom{k}{n} = \frac{k(k-1)(k-2)(k-3)\cdots(k-n+1)}{n!}$$

where k is a real number, n is a positive integer, and

$$\binom{k}{0} = 1.$$

89. $\binom{5}{3}$ **90.** $\binom{-2}{2}$

91. $\binom{0.5}{4}$ **92.** $\binom{-1/3}{5}$

93. Writing a Power Series Write the power series for $(1+x)^k$ in terms of binomial coefficients.

94. Proof Prove that e is irrational. [*Hint:* Assume that $e = p/q$ is rational (p and q are integers) and consider

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots.]$$

95. Using Fibonacci Numbers Show that the Maclaurin series for the function

$$g(x) = \frac{x}{1-x-x^2}$$

is

$$\sum_{n=1}^{\infty} F_n x^n$$

where F_n is the nth Fibonacci number with $F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$, for $n \geq 3$.

(*Hint:* Write

$$\frac{x}{1-x-x^2} = a_0 + a_1x + a_2x^2 + \cdots$$

and multiply each side of this equation by $1 - x - x^2$.)

PUTNAM EXAM CHALLENGE

96. Assume that $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all x on an interval of length at least 2. Show that $|f'(x)| \leq 2$ on the interval.

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Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Listing the Terms of a Sequence **In Exercises 1–4, write the first five terms of the sequence.**

1. $a_n = 5^n$

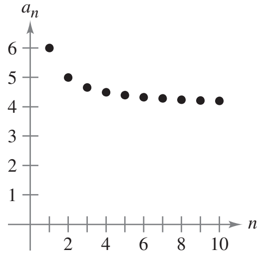
2. $a_n = \dfrac{3^n}{n!}$

3. $a_n = \left(-\dfrac{1}{4}\right)^n$

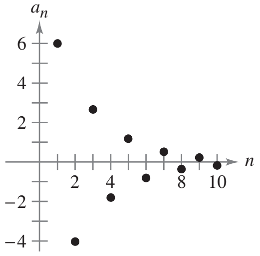
4. $a_n = \dfrac{2n}{n + 5}$

Matching **In Exercises 5–8, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

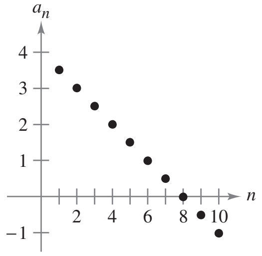
(a)



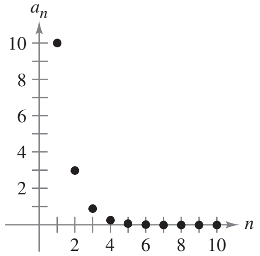
(b)



(c)



(d)



5. $a_n = 4 + \dfrac{2}{n}$

6. $a_n = 4 - \dfrac{1}{2}n$

7. $a_n = 10(0.3)^{n-1}$

8. $a_n = 6\left(-\frac{2}{3}\right)^{n-1}$

Finding the Limit of a Sequence **In Exercises 9 and 10, use a graphing utility to graph the first 10 terms of the sequence. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.**

9. $a_n = \dfrac{5n + 2}{n}$

10. $a_n = \sin \dfrac{n\pi}{2}$

Determining Convergence or Divergence **In Exercises 11–18, determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.**

11. $a_n = \left(\dfrac{2}{5}\right)^n + 5$

12. $a_n = 3 - \dfrac{2}{n^2 - 1}$

13. $a_n = \dfrac{n^3 + 1}{n^2}$

14. $a_n = \dfrac{1}{\sqrt{n}}$

15. $a_n = \dfrac{n}{n^2 + 1}$

16. $a_n = \dfrac{n}{\ln n}$

17. $a_n = \sqrt{n + 1} - \sqrt{n}$

18. $a_n = \dfrac{\sin \sqrt{n}}{\sqrt{n}}$

Finding the nth Term of a Sequence **In Exercises 19–22, write an expression for the nth term of the sequence. (There is more than one correct answer.)**

19. $3, 8, 13, 18, 23, \ldots$

20. $-5, -2, 3, 10, 19, \ldots$

21. $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{7}, \dfrac{1}{25}, \dfrac{1}{121}, \ldots$

22. $\dfrac{1}{2}, \dfrac{2}{5}, \dfrac{3}{10}, \dfrac{4}{17}, \ldots$

23. Compound Interest A deposit of \$8000 is made in an account that earns 5% interest compounded quarterly. The balance in the account after n quarters is

$$A_n = 8000\left(1 + \frac{0.05}{4}\right)^n, \quad n = 1, 2, 3, \ldots.$$

(a) Compute the first eight terms of the sequence $\{A_n\}$.

(b) Find the balance in the account after 10 years by computing the 40th term of the sequence.

24. Depreciation A company buys a machine for \$175,000. During the next 5 years, the machine will depreciate at a rate of 30% per year. (That is, at the end of each year, the depreciated value will be 70% of what it was at the beginning of the year.)

(a) Find a formula for the nth term of the sequence that gives the value V of the machine t full years after it was purchased.

(b) Find the depreciated value of the machine at the end of 5 full years.

Finding Partial Sums **In Exercises 25 and 26, find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5.**

25. $3 + \dfrac{3}{2} + 1 + \dfrac{3}{4} + \dfrac{3}{5} + \cdots$

26. $-\dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \dfrac{1}{16} - \dfrac{1}{32} + \cdots$

Numerical, Graphical, and Analytic Analysis **In Exercises 27–30, (a) use a graphing utility to find the indicated partial sum S_n and complete the table, and (b) use a graphing utility to graph the first 10 terms of the sequence of partial sums.**

n	5	10	15	20	25
S_n					

27. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^{n-1}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n + 1)}$

Finding the Sum of a Convergent Series **In Exercises 31–34, find the sum of the convergent series.**

31. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n$

32. $\displaystyle\sum_{n=0}^{\infty}\frac{3^{n+2}}{7^n}$

33. $\displaystyle\sum_{n=1}^{\infty}\left[(0.6)^n + (0.8)^n\right]$

34. $\displaystyle\sum_{n=0}^{\infty}\left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)}\right]$

Using a Geometric Series **In Exercises 35 and 36, (a) write the repeating decimal as a geometric series, and (b) write its sum as the ratio of two integers.**

35. $0.\overline{09}$

36. $0.\overline{64}$

Using Geometric Series or the *n*th-Term Test **In Exercises 37–40, use geometric series or the *n*th-Term Test to determine the convergence or divergence of the series.**

37. $\displaystyle\sum_{n=0}^{\infty}(1.67)^n$

38. $\displaystyle\sum_{n=0}^{\infty}(0.36)^n$

39. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n n}{\ln n}$

40. $\displaystyle\sum_{n=0}^{\infty}\frac{2n+1}{3n+2}$

41. **Distance** A ball is dropped from a height of 8 meters. Each time it drops h meters, it rebounds $0.7h$ meters. Find the total distance traveled by the ball.

42. **Compound Interest** A deposit of \$125 is made at the end of each month for 10 years in an account that pays 3.5% interest, compounded monthly. Determine the balance in the account at the end of 10 years. (*Hint:* Use the result of Section 9.2, Exercise 84.)

Using the Integral Test or a *p*-Series **In Exercises 43–48, use the Integral Test or a *p*-series to determine the convergence or divergence of the series.**

43. $\displaystyle\sum_{n=1}^{\infty}\frac{2}{6n+1}$

44. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n^3}}$

45. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$

46. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{5^n}$

47. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n}\right)$

48. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^4}$

Using the Direct Comparison Test or the Limit Comparison Test **In Exercises 49–54, use the Direct Comparison Test or the Limit Comparison Test to determine the convergence or divergence of the series.**

49. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt[3]{n}-1}$

50. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3+3n}}$

51. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+2n}}$

52. $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)}$

53. $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$

54. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n-5}$

Using the Alternating Series Test **In Exercises 55–60, use the Alternating Series Test, if applicable, to determine the convergence or divergence of the series.**

55. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}$

56. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)}{n^2+1}$

57. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2-3}$

58. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n+1}$

59. $\displaystyle\sum_{n=4}^{\infty}\frac{(-1)^n n}{n-3}$

60. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\ln n^3}{n}$

Using the Ratio Test or the Root Test **In Exercises 61–66, use the Ratio Test or the Root Test to determine the convergence or divergence of the series.**

61. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n-1}{2n+5}\right)^n$

62. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n}{7n-1}\right)^n$

63. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$

64. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{e^n}$

65. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^3}$

66. $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)}$

Numerical, Graphical, and Analytic Analysis **In Exercises 67 and 68, (a) verify that the series converges, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums, and (d) use the table to estimate the sum of the series.**

n	5	10	15	20	25
S_n					

67. $\displaystyle\sum_{n=1}^{\infty}n\left(\frac{3}{5}\right)^n$

68. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^3+5}$

Finding a Maclaurin Polynomial **In Exercises 69 and 70, find the *n*th Maclaurin polynomial for the function.**

69. $f(x) = e^{-2x}, \quad n = 3$

70. $f(x) = \cos \pi x, \quad n = 4$

Finding a Taylor Polynomial **In Exercises 71 and 72, find the third-degree Taylor polynomial centered at *c*.**

71. $f(x) = e^{-3x}, \quad c = 0$

72. $f(x) = \tan x, \quad c = -\dfrac{\pi}{4}$

Finding a Degree **In Exercises 73 and 74, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of *x* to be less than 0.001.**

73. $\cos(0.75)$

74. $e^{-0.25}$

Finding the Interval of Convergence **In Exercises 75–80, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)**

75. $\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$

76. $\sum_{n=0}^{\infty} (5x)^n$

77. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(n+1)^2}$

78. $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$

79. $\sum_{n=0}^{\infty} n!(x-2)^n$

80. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n}$

Finding Intervals of Convergence **In Exercises 81 and 82, find the intervals of convergence of (a) $f(x)$, (b) $f'(x)$, (c) $f''(x)$, and (d) $\int f(x)\,dx$. Include a check for convergence at the endpoints of the interval.**

81. $f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$

82. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-4)^n}{n}$

Differential Equation **In Exercises 83 and 84, show that the function represented by the power series is a solution of the differential equation.**

83. $y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n (n!)^2}$

$x^2y'' + xy' + x^2y = 0$

84. $y = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n\, n!}$

$y'' + 3xy' + 3y = 0$

Finding a Geometric Power Series **In Exercises 85 and 86, find a geometric power series, centered at 0, for the function.**

85. $g(x) = \frac{2}{3-x}$

86. $h(x) = \frac{3}{2+x}$

Finding a Power Series **In Exercises 87 and 88, find a power series for the function, centered at c, and determine the interval of convergence.**

87. $f(x) = \frac{6}{4-x}, \quad c = 1$

88. $f(x) = \frac{1}{3-2x}, \quad c = 0$

Finding the Sum of a Series **In Exercises 89–94, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.**

89. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n\, n}$

90. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n\, n}$

91. $\sum_{n=0}^{\infty} \frac{1}{2^n\, n!}$

92. $\sum_{n=0}^{\infty} \frac{2^n}{3^n\, n!}$

93. $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!}$

94. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!}$

Finding a Taylor Series **In Exercises 95–102, use the definition of Taylor series to find the Taylor series, centered at c, for the function.**

95. $f(x) = \sin x, \quad c = \frac{3\pi}{4}$

96. $f(x) = \cos x, \quad c = -\frac{\pi}{4}$

97. $f(x) = 3^x, \quad c = 0$

98. $f(x) = \csc x, \quad c = \frac{\pi}{2}$ (first three terms)

99. $f(x) = \frac{1}{x}, \quad c = -1$

100. $f(x) = \sqrt{x}, \quad c = 4$

101. $g(x) = \sqrt[5]{1+x}, \quad c = 0$

102. $h(x) = \frac{1}{(1+x)^3}, \quad c = 0$

103. **Forming Maclaurin Series** Determine the first four terms of the Maclaurin series for e^{2x}

(a) by using the definition of the Maclaurin series and the formula for the coefficient of the nth term, $a_n = f^{(n)}(0)/n!$.

(b) by replacing x by $2x$ in the series for e^x.

(c) by multiplying the series for e^x by itself, because $e^{2x} = e^x \cdot e^x$.

104. **Forming Maclaurin Series** Determine the first four terms of the Maclaurin series for $\sin 2x$

(a) by using the definition of the Maclaurin series and the formula for the coefficient of the nth term, $a_n = f^{(n)}(0)/n!$.

(b) by replacing x by $2x$ in the series for $\sin 2x$.

(c) by multiplying 2 by the series for $\sin x$ by the series for $\cos x$, because $\sin 2x = 2 \sin x \cos x$.

Finding a Maclaurin Series **In Exercises 105–108, find the Maclaurin series for the function. Use the table of power series for elementary functions on page 670.**

105. $f(x) = e^{6x}$

106. $f(x) = \ln(x-1)$

107. $f(x) = \sin 2x$

108. $f(x) = \cos 3x$

Finding a Limit **In Exercises 109 and 110, use the series representation of the function f to find $\lim_{x \to 0} f(x)$ (if it exists).**

109. $f(x) = \frac{\arctan x}{\sqrt{x}}$

110. $f(x) = \frac{\arcsin x}{x}$